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# THE TOWER OF $n$-GROUPOIDS AND THE LONG COHOMOLOGY SEQUENCE

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Given an exact category $\mathbb{E}$, we associate to it a fibration $c$ above $\mathbb{E}$ such that, for each object $X$ of $\mathbb{E}$, the fiber $c[x]$ is again exact. If, moreover, $A$ is an internal abelian group in $\mathbb{E}$, it determines a family of abelian groups $A_x$ in the fibres $c[x]$, such that the group $H^1(\mathbb{E},A)$ is the colimit of the $H^0(c[x],A_x)$. This remark allows us to define iteratively $H^{n+1}(\mathbb{E},A)$ as the colimit of the $H^n(c[x],A_x)$. These groups are shown to have the property of the long cohomology sequence. When $\mathbb{E}=\mathrm{Ab}$, the construction coincides, up to isomorphism, with Yoneda's classical description of $\mathrm{Ext}^n$. When $\mathbb{E}=\mathrm{Grp}$, it coincides with the cohomology groups of a group in the sense of Eilenberg–Mac Lane.

## Introduction

An interpretation of the first and second cohomology groups of a group, in the sense of Eilenberg–Mac Lane [17], was given in [2], in terms of connected components of a lax limit and a 3-dimensional lax limit. It was just a remark, bringing a geometrical flavour to the abstract formulae of crossed homomorphisms, principal homomorphisms and factor sets. At that time, the investigation of general $n$-categories seemed to be premature, making a similar interpretation for higher cohomology groups impossible. Just such an interpretation is the aim of this paper.

Indeed, since that time, much work has been done in this area. First there were the papers of A. and C. Ehresmann [15] about $n$-fold categories, which are even more general than $n$-categories, and a paper on their recent use in $K$-theory by Shimakawa [28]. There were also the papers by Brown and Higgins [9–11], concerning the generalization of the Seifert–Van Kampen theorem: an important step of their work consisted in the proof of the equivalence between the category of crossed complexes and the category of $\infty$-groupoids. Finally, there was the work of Loday [25] in homotopy theory, using an equivalent of internal $n$-fold categories in the category Grp of abstract groups.

All this work seemed to be sufficient reason to undertake the systematic investigation of internal $n$-categories [3, 6–8], a different approach being taken and parallel progress being made by Street and the Australian school [29], and also to return to the problem of realizing cohomology groups (and not only abstract groups) by means of $n$-categories or, better, $n$-groupoids.

In order to see this, let us remember that, if $\mathbb{E}$ is a Barr exact category and $A$ an abelian group in $\mathbb{E}$, the group $H^0(\mathbb{E},A)$ is defined as the group $\mathbb{E}(1,A)$ of global elements of $A$, and the group $H^1(\mathbb{E},A)$ as the group of connected components of the monoidal groupoid $\mathrm{Tors}(\mathbb{E},A)$ of $A$-torsors, that is to say, principal $A$-actions on objects of $\mathbb{E}$ having a global support (see for instance [1]). Though there were other presentations of higher order cohomology groups, for instance by means of special kinds of simplicial Kan fibrations by Duskin [14] and Glenn [19], such a presentation of cohomology groups by means of group actions, in the usual sense, actually stops at level one with the well-known six-term exact sequence.

Now, let us denote by $\mathrm{Grd}\,\mathbb{E}$ the category of internal groupoids in $\mathbb{E}$ and by $c$ the forgetful functor $c:\mathrm{Grd}\,\mathbb{E}\to\mathbb{E}$, associating to each groupoid $X_1$ its object of objects $X_0$. It is in fact a fibration, whose fibers $c[M]$, for each object $M$ in $\mathbb{E}$, are again Barr exact. Furthermore, if $A$ is an abelian group in $\mathbb{E}$, an abelian group being a group in the category of groups in $\mathbb{E}$, it obviously determines an abelian group $K_1(A)$ in the fiber $c[1]$ and therefore, by change of base along the terminal map $M\to 1$, an abelian group $M^*(K_1(A))$ in each fiber $c[M]$. The starting point of this work is the following remark: the abelian group $H^1(\mathbb{E},A)$ is nothing but the colimit of the abelian groups $H^0(c[M],M^*(K_1A))$, the objects $M$ having a global support. This point is simply based on the fact that any functor between internal groupoids in $\mathbb{E}$ has an associated discrete fibration.

It is therefore natural to investigate whether the colimit of the group $H^1(c[M], M^*(K_1A))$ (which are the $H^1$ of the fibers of $c$) is, in the classical examples, a realization of the second cohomology group, and, more generally, whether that is the case for the higher order groups (let us denote them beforehand by $H^{n+1}(\mathbb{E},A)$) defined iteratively by the colimits of the $H^n(c[M],M^*(K_1A))$ (which are the $H^n$ of the fibers of $c$).

Now, what are the objects of $H^2(\mathbb{E},A)$? According to our initial remark, they are described by classes of internal groupoids in the fibers of the fibration $c$. That is to say, strictly speaking, internal 2-groupoids in $\mathbb{E}$. More generally, the objects of $H^n(\mathbb{E},A)$ are described by classes of what appear to be exactly internal $n$-groupoids in $\mathbb{E}$. When $\mathbb{E}=\mathbb{A}$ is an abelian category, this description of the $H^n$ coincides, up to isomorphism, with Yoneda's classical description of $\mathrm{Ext}^n$. There is, indeed, another possible denormalization theorem for abelian chain complexes [4]: that is, for each integer $n$, a natural (in $n$) equivalence between the category $n$-$\mathrm{Grd}\,\mathbb{A}$ of internal $n$-groupoids in $\mathbb{A}$ and the category $C^n(\mathbb{A})$ of abelian chain complexes of length $n$. Similarly, when $\mathbb{E}$ is the category Grp of abstract groups, this description is equivalent to those given by Holt [22] and Huebschmann [23] of the homology groups of a group in the sense of Eilenberg–Mac Lane, by means of crossed $n$-fold extensions.

The aim of this paper, summarized in [5], is thus to describe precisely these groups $H^n(\mathbb{E},A)$ and to show that they have the property of the long cohomology sequence. The introduction of $n$-groupoids in the realization of cohomology groups does not seem to be a mere gadget. On the one hand, it appears to be the most direct way to extend the classical description of the first cohomology group by means of principal actions to the higher order cohomology groups. On the other hand, while, in the abelian situation, the usual Dold–Kan equivalence between abelian chain complexes and simplicial abelian groups does not strictly exchange chain homotopies with simplicial homotopies, the new equivalences between $C^n(\mathbb{A})$ and $n$-Grd $\mathbb{A}$ do strictly exchange chain homotopies with what is known as (higher order) pseudonatural transformations. In this sense, the two notions of $n$-complexes and internal $n$-groupoids seem to be more closely connected than the notions of complexes and simplicial objects.

*Main definitions and results*

Now to be more explicit and before going into detail, let us recall the main definitions concerning internal $n$-groupoids and let us introduce briefly the cohomology groups.

*A. Internal n-categories and n-groupoids*

Let $\mathbb{V}$ be a left exact category. An internal category in $\mathbb{V}$ is a diagram $X_1$ in $\mathbb{V}$,

$$\begin{tikzcd} X_0 \arrow[r, "s_0"] & mX_1 \arrow[l, shift right=3, "d_0"'] \arrow[l, shift left=3, "d_1"] & m_2X_1 \arrow[l, shift right=3, "d_0"'] \arrow[l, "d_1"] \arrow[l, shift left=3, "d_2"] \end{tikzcd}$$

such that $m_2X_1$ is the vertex of the pullback of $d_0$ along $d_1$, satisfying the usual unitary and associativity axioms. An internal functor is just a natural transformation between such diagrams. Let Cat $\mathbb{V}$ denote the category of internal categories in $\mathbb{V}$ and $(\ )_0$ the forgetful left exact functor Cat $\mathbb{V} \to \mathbb{V}$ associating $X_0$ to $X_1$. It has a fully faithful right adjoint $G_1$.

An internal category is discrete when $s_0$ is an isomorphism. It is a groupoid if the following diagram is a pullback:

$$\begin{tikzcd} mX_1 \arrow[d, "d_1"'] & m_2X_1 \arrow[l, "d_1"'] \arrow[d, "d_2"] \\ X_0 & mX_1. \arrow[l, "d_1"] \end{tikzcd}$$

Let Grd $\mathbb{V}$ denote the full subcategory of Cat $\mathbb{V}$ whose objects are the internal groupoids and again $(\ )_0$ the forgetful functor. Whence the following situation:

$$\begin{tikzcd} \mathrm{Grd}\,\mathbb{V} \arrow[r, shift left, "(\ )_0"] & \mathbb{V}. \arrow[l, shift left, "G_1"] \end{tikzcd}$$

Now suppose we are in the general similar situation (called the basic situation)

$$\mathbb{V} \underset{d}{\overset{c}{\rightleftarrows}} \mathbb{W}$$

with $\mathbb{V}$ left exact, $c$ left exact and $d$ a fully faithful right adjoint of $c$. We denote by $\mathrm{Cat}_c\mathbb{V}$ the full subcategory of $\mathrm{Cat}\,\mathbb{V}$ whose objects, called $c$-discrete categories, are the internal categories in $\mathbb{V}$ mapped by $c$ on a discrete category in $\mathbb{W}$. The latter is the case if and only if $c(s_0)$ is an isomorphism. Denote by $\mathrm{Grd}_c\mathbb{V}$ the full category of $\mathrm{Cat}_c\mathbb{V}$ whose objects are groupoids. Let $c_0 : \mathrm{Grd}_c\mathbb{V} \to \mathbb{V}$ be the forgetful functor. It is left exact and has a fully faithful right adjoint $G_1$, given for any object $V$ in $\mathbb{V}$ by the kernel equivalence relation of $V \to dcV$:

$$\begin{tikzcd} dcV & V \arrow[l, dashed] \arrow[r] & V \underset{c}{\times} V \arrow[l, shift right=2, "p_0"'] \arrow[l, shift left=2, "p_1"] & V \underset{c}{\times} V \underset{c}{\times} V. \arrow[l, shift right=3, "p_0"'] \arrow[l, "p_1"] \arrow[l, shift left=3, "p_2"] \end{tikzcd}$$

Consequently we are again in the basic situation:

$$\mathrm{Grd}_c\mathbb{V} \underset{G_1}{\overset{c_0}{\rightleftarrows}} \mathbb{V}$$

and in the position to iterate the process.

Let us denote the $n$th iterated step of this basic construction in the following way:

$$n\text{-}\mathrm{Grd}_c\mathbb{V} \underset{G_n}{\overset{c_{n-1}}{\rightleftarrows}} (n-1)\text{-}\mathrm{Grd}_c\mathbb{V}.$$

An object of $n$-$\mathrm{Grd}_c\mathbb{V}$ is called a $c$-discrete $n$-groupoid, and a morphism an internal $n$-functor. Now starting from a left exact category $\mathbb{E}$ and from the following basic situation:

$$\mathbb{E} \underset{1}{\rightleftarrows} \mathbb{1}$$

where 1 is a terminal object in $\mathbb{E}$, we obtain the following tower

$$\ldots\; n\text{-}\mathrm{Grd}\,\mathbb{E} \underset{G_n}{\overset{(\;)_{n-1}}{\rightleftarrows}} (n-1)\text{-}\mathrm{Grd}\,\mathbb{E} \;\ldots\; 2\text{-}\mathrm{Grd}\,\mathbb{E} \underset{G_2}{\overset{(\;)_1}{\rightleftarrows}} \mathrm{Grd}\,\mathbb{E} \underset{G_1}{\overset{(\;)_0}{\rightleftarrows}} \mathbb{E} \rightleftarrows \mathbb{1}$$

whose $n$th term is called the category of internal $n$-groupoids in $\mathbb{E}$.

*B. The first cohomology group relative to c*

Suppose we are again in the basic situation with, moreover, $c$ Barr-exact [4] meaning that each $c$-discrete equivalence relation in $\mathbb{V}$ is effective and has a universal cokernel. Let $A$ be an abelian group in $\mathbb{V}$. A left $A$-object is well known to be an object $V$ in $\mathbb{V}$, together with a left action $\upsilon : A \times V \to V$ with $\upsilon \cdot [0, V] = V$ and $\upsilon \cdot (A \times \upsilon) = \upsilon \cdot (+ \times V)$, where $V$ denotes $1_V$ for short, a morphism between $A$-objects being a morphism which commutes with the given actions. Let $L\mathbb{V}[A]$ denote the category of left $A$-objects in $\mathbb{V}$.

Now let $K_1(A)$ be the following internal groupoid (actually a group) in $\mathbb{V}$:

$$\begin{tikzcd} K_1(A): 1 \arrow[r, "0"] & A \arrow[l, shift right=2] \arrow[l, shift left=2] & A\times A. \arrow[l, shift right=2, "p_0"'] \arrow[l, "+"] \arrow[l, shift left=2, "p_1"] \end{tikzcd}$$

It is well known that the category $L\mathbb{V}[A]$ is equivalent to the category $FD/K_1(A)$ of internal discrete fibrations above $K_1(A)$, this equivalence associating to $(V, \upsilon)$ the following internal discrete fibration:

$$\begin{tikzcd} F_1(V,\upsilon) : \arrow[d, "f_1(V,\upsilon)"'] & V \arrow[r] \arrow[d] & A\times V \arrow[l, shift right=2, "\upsilon"'] \arrow[l, shift left=2, "p_V"] \arrow[d, "p_A"] & A\times A\times V \arrow[l, shift right=2, "A\times\upsilon"'] \arrow[l, "+\times V"] \arrow[l, shift left=2, "p_1\times V"] \arrow[d, "p_{A\times A}"] \\ K_1(A) : & 1 \arrow[r] & A \arrow[l, shift right=2] \arrow[l, shift left=2] & A\times A. \arrow[l, shift right=2] \arrow[l] \arrow[l, shift left=2] \end{tikzcd}$$

Later on we shall often identify the left $A$-object $(V, \upsilon)$ with the discrete fibration $f_1(V, \upsilon)$.

**Definition 1.** The group $A$ is said to be $c$-trivial if $c(A)$ is equal to 1, or equivalently if $K_1(A)$ is $c$-discrete.

**Proposition 2.** *If $A$ is c-trivial, then the groupoid $F_1(V, \upsilon)$ is c-discrete.*

**Proof.** The morphism $c(p_V)$ is the inverse of $c[0, V]$ since $c$ is left exact and $A$ $c$-trivial. $\square$

Now this groupoid being $c$-discrete and $F_0(V, \upsilon) = V$, there is an internal functor

$$\varphi_1 : F_1(V, \upsilon) \to G_1(V).$$

**Definition 3.** The left object $(V, \upsilon)$ is said to be $c$-principal when this functor $\varphi_1$ is an isomorphism, or equivalently when the following map in $\mathbb{V}$:

$$[\upsilon, p_V] : A \times V \to V \times_c V$$

is an isomorphism.

Let $c$-$P\mathbb{V}[A]$ denote the category of $c$-principal $A$-objects in $\mathbb{V}$.

**Definition 4.** An object $V$ in $\mathbb{V}$ is said to have a global $c$-support if the map $V \to dcV$ is a regular epimorphism.

**Definition 5.** A $c$-torsor is a $c$-principal $A$-object $(V, \upsilon)$ with $V$ having a global $c$-support.

Given a $c$-trivial abelian group $A$, we define Tors$(c, A)$ as the full subcategory of $c$-$P\mathbb{V}[A]$ whose objects are the $c$-torsors. Now, the functor $c$ being Barr-exact, there is on Tors$(c, A)$ a tensor product, defined exactly as in the classical situation.

The group of connected components of Tors$(c, A)$ is denoted $H^1(c, A)$ and called the first cohomology group relative to $c$.

*C. The higher order cohomology groups of c*

If $c$ is Barr-exact, then $c_0$ is again Barr-exact (see [3]). The object $K_1(A)$ is actually an abelian group in $\mathrm{Grd}_c\mathbb{V}$, since, $A$ being abelian, the morphism $+: A \times A \to A$ determines an internal functor $+_1 : K_1(A) \times K_1(A) \to K_1(A)$.

Moreover, this abelian group is clearly $c_0$-trivial. Consequently we are again in the previous situation with $c_0 : \mathrm{Grd}_c\mathbb{V} \to \mathbb{V}$ Barr-exact and $K_1(A)$ a $c_0$-trivial abelian group in $\mathrm{Grd}_c\mathbb{V}$. So we have again a monoidal category Tors$(c_0, K_1(A))$.

**Definition 6.** A $c$-discrete groupoid $X_1$ is said to be $c$-aspherical when $X_1$ has a global $c_0$-support and $X_0$ a global $c$-support.

Let 2-Tors$(c, A)$ be the full subcategory of Tors$(c_0, K_1(A))$ whose objects $(X_1, x_1)$ are such that $X_1$ is $c$-aspherical. The tensor product of Tors$(c_0, K_1(A))$ is stable on 2-Tors$(c, A)$. The group of connected components of 2-Tors$(c, A)$ is denoted $H^2(c, A)$ and called the second cohomology group relative to $c$.

More generally, the functor $c_{n-1} : n\text{-}\mathrm{Grd}_c\mathbb{V} \to (n-1)\text{-}\mathrm{Grd}_c\mathbb{V}$ is again Barr exact. We have, in $n$-Grd $\mathbb{V}$, a $c_{n-1}$-trivial abelian group defined by the formula $K_n(A) = K_1(K_{n-1}(A))$.

**Definition 7.** An object $X_n$ of $n\text{-}\mathrm{Grd}_c\mathbb{V}$ is said to be $c$-aspherical if $X_n$ has a global $c_{n-1}$-support and, moreover, $X_{n-1}$ is $c$-aspherical.

Let $(n+1)$-Tors$(c, A)$ be the full subcategory of Tors$(c_{n-1}, K_n(A))$ whose objects $(X_n, x_n)$ are such that $X_n$ is aspherical. The tensor product of Tors$(c_{n-1}, K_n(A))$ is stable on $(n+1)$-Tors$(c, A)$. The group of connected components of $(n+1)$-Tors$(c, A)$ is denoted $H^{n+1}(c, A)$ and called the $(n+1)$th cohomology group relative to $c$.

This paper will be mainly devoted to show that these $H^n(c, A)$ have the property of the cohomology long exact sequence.

Here is the organization of this paper:

1. Internal $n$-groupoids;
2. The cohomology groups relative to a fibration;
3. The exactness property of the long cohomology sequence;
4. The classical examples.

## 1. Internal $n$-groupoids

In this section, we shall recall the main results about the basic situation and the internal $n$-groupoids.

### *1.1. The basic situation*

A functor $c$, together with the whole basic situation, will be called a left exact fibered reflection [3], such a definition being justified by the fact that $c$ is just a left exact functor and a fibered reflection that is, up to equivalence, a fibration with a terminal object in each fiber.

A $c$-Cartesian morphism is nothing but a morphism $f: V \to V'$ in $\mathbb{V}$ such that the following diagram is a pullback:

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d] & V' \arrow[d] \\ dcV \arrow[r, "dcf"'] & dcV'. \end{tikzcd}$$

The class $c$-Cart of $c$-Cartesian morphisms is obviously stable under pullback and composition. It contains the isomorphisms. Furthermore, if $g$ and $g \cdot f$ are in $c$-Cart, then $f$ is in $c$-Cart. Finally, a morphism $dh$, for any $h$ in $\mathbb{W}$, is trivially $c$-Cartesian.

Now given a class of morphisms $\Xi$ in $\mathbb{V}$, let us denote by $\Xi^{\perp}$ the class of morphisms $g$ in $\mathbb{V}$, satisfying the diagonality condition [12, 18, 30]: for any commutative square

$$\begin{tikzcd} S \arrow[r] \arrow[d, "g"'] & X \arrow[d, "f"] \\ T \arrow[r] \arrow[ur, dashed] & y \end{tikzcd}$$

with $f$ in $\Xi$, there is a unique dotted arrow making the two triangles commutative.

The class $(c\text{-Cart})^{\perp}$ is just the class $c$-Inv of $c$-invertible morphisms, that is morphisms whose image by $c$ is invertible. The class $c$-Inv is stable under pullback and composition. It contains the isomorphisms. Furthermore, if any two of the three morphisms $f$, $g$ and $g \cdot f$ are in $c$-Inv, then the third one is in $c$-Inv.

It is clear that a morphism which is at the same time $c$-Cartesian and $c$-invertible, is an isomorphism. Any morphism $f$ in $\mathbb{V}$ has a unique, up to isomorphism, decomposition $f = f^{c} \cdot f^{i}$ with $f^{i}$ $c$-invertible and $f^{c}$ $c$-Cartesian, given by the following diagram in which the square $(*)$ is a pullback:

$$\begin{tikzcd} V \arrow[rr, "f"] \arrow[dr, "f^{i}"'] \arrow[dd] & & V' \arrow[dd] \\ & \bullet \arrow[ur, "f^{c}"'] \arrow[dl] \arrow[r, phantom, "(*)"] & {} \\ dcV \arrow[rr, "dcf"'] & & dcV'. \end{tikzcd}$$

Finally, a commutative square with a pair of parallel edges in $c$-Cart and the other one in $c$-Inv is certainly a pullback.

**Example.** In the case $\mathbb{E}$ left exact and $c = (\ )_0 : \mathrm{Grd}\,\mathbb{E} \to \mathbb{E}$, a $(\ )_0$-Cartesian morphism is an internally fully faithful functor and a $(\ )_0$-invertible one is an internal functor which is 'bijective on objects'.

### *1.2. Barr-exact fibered reflection*

The fibered reflection $c$ is said to be Barr-exact if, moreover, any $c$-invertible equivalence relation in $\mathbb{V}$ has a quotient (a coequalizer making this equivalence relation effective) which is universal. Clearly such a quotient is $c$-invertible.

**Remark.** A fibered reflection is Barr-exact if and only if its associated fibration (up to equivalence) has its fibers and its change of base functors Barr-exact [3].

**Example.** When $\mathbb{E}$ is Barr-exact and left exact, then the functor $(\ )_0 : \mathrm{Grd}\,\mathbb{E} \to \mathbb{E}$ is a Barr-exact fibered reflection.

An object $V$ in $\mathbb{V}$ is said to have a global $c$-support when the terminal map in the fiber: $V \to dcV$ is a $c$-invertible regular epimorphism. It is clear that, if $f: V \to V'$ is $c$-Cartesian and $V'$ has a global $c$-support, then $V$ has a global $c$-support. Furthermore, $c$ being left exact, the product of two objects $U$ and $V$ with global $c$-supports has a global $c$-support.

**Definition 8.** A morphism $f: V \to V'$ in $\mathbb{V}$ is called $c$-faithful if its $c$-invertible part $f^{\mathrm{i}}$ is a monomorphism and $c$-full if $f^{\mathrm{i}}$ is a regular epimorphism.

Let $\Sigma$ denote the class of $c$-full morphisms. It is stable under pullback and composition. It clearly contains $c$-Cart.

Thus an object $V$ in $\mathbb{V}$ has global $c$-support if and only if its terminal map in $\mathbb{V}$, $V \to 1$, is $c$-full. Consequently, when $f: U \to V$ is in $\Sigma$ and $V$ has a global $c$-support, then $U$ has a global $c$-support.

**Example.** As expected from this terminology, a $(\ )_0$-faithful morphism in $\mathrm{Grd}\,\mathbb{E}$ (resp. $(\ )_0$-full) is simply an internally faithful (resp. full) functor.

### *1.3. The c-discrete groupoids*

Let us consider now the following fibered reflection:

$$\mathrm{Grd}_c\mathbb{V} \underset{G_1}{\overset{c_0}{\rightleftarrows}} \mathbb{V}.$$

There is again a canonical decomposition dealing with $c_0$-invertible and $c_0$-Cartesian morphisms.

Now, if $c$ is Barr-exact, then $c_0$ is again Barr-exact [3]. We shall say that a $c$-discrete groupoid $X_1$ is $c$-aspherical when $X_1$ has a global $c_0$-support and $X_0$ a global $c$-support. If $f_1: X_1 \to Y_1$ is a morphism between aspherical $c$-groupoids, then the vertex $Z_1$ of its canonical decomposition,

$$X_1 \xrightarrow{f_1^{\mathrm{i}}} Z_1 \xrightarrow{f_1^{\mathrm{c}}} Y_1,$$

is aspherical: $Z_1$ has a global $c_0$-support since $f_1^{\mathrm{c}}$ is Cartesian and $Y_1$ has a global $c_0$-support. Moreover, $Z_0$, being isomorphic to $X_0$, has a global $c$-support.

*The class $\Sigma_1$*

There is, in $\mathrm{Grd}_c\mathbb{V}$, a class of morphisms which will be very important for us, namely the class $\Sigma_1$ of morphisms $f_1: X_1 \to Y_1$ such that $f_1$ is $c_0$-full and $f_0: X_0 \to Y_0$ is $c$-full. This class $\Sigma_1$ is stable under pullback and composition. It contains the isomorphisms.

A $c$-discrete groupoid $Y_1$ is then aspherical when its terminal map $Y_1 \to 1$ is in $\Sigma_1$. Consequently, if $f_1: X_1 \to Y_1$ is in $\Sigma_1$ and $Y_1$ is aspherical, then $X_1$ is aspherical.

At last, it is easy to check that if $f_1$ is in $\Sigma_1$, then $mf_1: mX_1 \to mY_1$ is again $c$-full.

*The functor $\pi_0$*

Actually the functor $c_0$ has also a left adjoint dis, where dis $V$ is the $c$-discrete groupoid with every structural map equal to $1_V$. When $c$ is Barr-exact, this functor dis has itself a left adjoint $\pi_0: \mathrm{Grd}_c\mathbb{V} \to \mathbb{V}$, which is a fibered reflection (see [7]) (obviously, no longer left exact).

## *1.4. The discrete fibrations and the final functors*

From now on, we shall suppose $c$ Barr-exact. Besides the $c_0$-Cartesian $c_0$-invertible decomposition, there is in $\mathrm{Grd}_c\mathbb{V}$ another significant factorization system.

Let us recall that an internal functor $f_1: X_1 \to Y_1$ is said to be a discrete fibration when the following square is a pullback:

$$\begin{tikzcd} mX_1 \arrow[r, "mf_1"] \arrow[d, "d_1"'] & mY_1 \arrow[d, "d_1"] \\ X_0 \arrow[r, "f_0"'] & Y_0. \end{tikzcd}$$

Let DF denote the class of discrete fibrations. It is stable under pullback and composition. It contains isomorphisms. If $g_1 \cdot f_1$ and $g_1$ are in DF, then $f_1$ is in DF. Any functor dis$f$ is a discrete fibration for any $f$ in $\mathbb{V}$. A functor in $\mathrm{DF}^{\perp}$ is called final.

When $\mathbb{E}$ is left exact and Barr-exact, it is shown in [6] that in Grd $\mathbb{E}$ every functor has a unique, up to isomorphism, factorization $f_1 = k_1 \cdot h_1$ with $k_1$ in DF and $h_1$ final. Furthermore, the final functors are stable under pullbacks along a discrete fibration. It is possible to check that exactly the same construction and the same result hold in $\mathrm{Grd}_c\mathbb{V}$ when $c$ is a left exact and Barr-exact fibered reflection.

Moreover, it is essential for us that this decomposition is stable under product.

Now, if $f: X \to Y$ is $c$-Cartesian in $\mathbb{V}$, then $G_1 f: G_1 X \to G_1 Y$ is a discrete fibration in $\mathrm{Grd}_c \mathbb{V}$. Conversely, when $G_1 f: G_1 X \to G_1 Y$ is a discrete fibration and the objects $X$ and $Y$ have a global $c$-support, then $f$ is $c$-Cartesian (see [6, Lemma 4]).

If $f$ is $c$-invertible, then $G_1 f$ is final if and only if $X$ and $Y$ have the same $c$-support. In particular, $G_1 f$ is final as soon as $X$ and $Y$ have a global $c$-support and $f$ is $c$-invertible. Finally, if a morphism $f_1: G_1 X \to Y_1$ is final, then there exists a $c$-invertible morphism $f: X \to Y$ such that $f_1 = G_1 f$. When, moreover, $X$ has a global $c$-support, this $Y$ has itself a global $c$-support.

On the other hand, the fibered reflection $\pi_0$ determines also a factorization system. It is shown in [6] that $\pi_0\text{-Cart} \subset \mathrm{DF}$ and consequently $\mathrm{DF}^{\perp} \subset \pi_0\text{-Inv}$. More precisely, it is possible to check that:

$$\mathrm{DF}^{\perp} = \pi_0\text{-Inv} \cap c_0\text{-Full}.$$

*1.5. The universal representor for natural transformations*

It is well known that the category $\mathrm{Cat}\,\mathbb{V}$ is actually underlying a 2-category. But, when the category $\mathbb{V}$ is left exact, this higher order structure, which will appear to be extremely important for the exactness property of the long sequence, can be entirely represented by 1-morphisms [20]. Indeed, for any category $X_1$ (resp. groupoid) there is a category (resp. groupoid) $\mathrm{Com}\, X_1$ together with two functors

$$\mathrm{Com}\, X_1 \overset{\sigma_1}{\underset{\tau_1}{\rightrightarrows}} X_1$$

such that any internal natural transformation

$$\begin{tikzcd} Y_1 \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & X_1 \end{tikzcd}$$

can be represented by a unique functor $Y_1 \to \mathrm{Com}\, X_1$.

If $X_1$ is $c$-discrete, then $\mathrm{Com}\, X_1$ is $c$-discrete.

Actually, there is a very strong connection between this 2-categorical structure and the fibration $c_0: \mathrm{Grd}_c \mathbb{V} \to \mathbb{V}$ which exempts us from further description.

**Proposition 9.** *A $c$-discrete category $X_1$ is a $c$-discrete groupoid if and only if $\sigma_1$ (resp. $\tau_1$)$: \mathrm{Com}\, X_1 \to X_1$ is $c_0$-Cartesian above $d_0$ (resp. $d_1$)$: mX_1 \to X_0$.*

**Proof.** See [7, Proposition 18 and Corollary]. □

This construction Com clearly extends to a left exact functor $\mathrm{Com}: \mathrm{Grd}_c \mathbb{V} \to \mathrm{Grd}_c \mathbb{V}$ and to natural transformations $\sigma_1$ and $\tau_1$. Furthermore, $\mathrm{Com}(G_1 V)$ is isomorphic to $G_1(V \times_c V)$. It is clear from Proposition 9 that Com preserves the $c_0$-Cartesian morphisms.

When $c$ is Barr-exact, if $X_1$ is aspherical, then Com $X_1$ is aspherical. Furthermore, if $f_1 : X_1 \to Y_1$ is in $\Sigma_1$, then Com $f_1$ is in $\Sigma_1$.

### *1.6. The c-discrete n-groupoids*

From the fact that the following fibered reflection:

$$n\text{-}\mathrm{Grd}_c\mathbb{V} \underset{G_n}{\overset{c_{n-1}}{\rightleftarrows}} (n-1)\text{-}\mathrm{Grd}_c\mathbb{V}$$

is again Barr-exact, we have also at this level, besides the $c_{n-1}$-invertible $c_{n-1}$-Cartesian decomposition, the final-discrete fibration decomposition. We have again a functor

$$\pi_{n-1} : n\text{-}\mathrm{Grd}_c\mathbb{V} \to (n-1)\text{-}\mathrm{Grd}_c\mathbb{V},$$

left adjoint to the functor dis.

*The aspherical objects and the class $\Sigma_n$*

The notion of aspherical objects in $n$-$\mathrm{Grd}_c\mathbb{V}$ is defined by induction from the notion of aspherical objects in $\mathrm{Grd}_c\mathbb{V}$:

A $c$-discrete $n$-groupoid $X_n$ is said to be $c$-aspherical if it has a global $c_{n-1}$-support and $X_{n-1}$ is $c$-aspherical. Then if $X_n \to Z_n \to Y_n$ is the canonical decomposition of $f_n$ associated to the fibration $c_{n-1}$ and if $X_n$ and $Y_n$ are $c$-aspherical, then $Z_n$ is $c$-aspherical: it has a global $c_{n-1}$-support since the right part is $c_{n-1}$-Cartesian and $Z_{n-1}$ is $c$-aspherical since the left part is $c_{n-1}$-invertible.

In the same way, there is a class $\Sigma_n$ defined by induction from $\Sigma_1$: an $n$-functor $f_n : X_n \to Y_n$ is in $\Sigma_n$ if it is $c_{n-1}$-full and $f_{n-1}$ is in $\Sigma_{n-1}$. The class $\Sigma_n$ is stable under pullback and composition. It contains isomorphisms.

An object $Y_n$ is then aspherical when its terminal map $Y_n \to 1$ is in $\Sigma_n$. Consequently, if $f_n : X_n \to Y_n$ is in $\Sigma_n$ and $Y_n$ is aspherical, then $X_n$ is aspherical.

Now $\sigma_n : \mathrm{Com}\, X_n \to X_n$ being $c_{n-1}$-Cartesian and its image by $(\ )_{n-1}$ being $d_0 : mX_n \to X_{n-1}$, that is a $c_{n-2}$-invertible split epimorphism, then $\sigma_n$ is in $\Sigma_n$. Consequently, Com $X_n$ is aspherical if $X_n$ is aspherical. Furthermore, if $f_n : Y_n \to Y_n$ is in $\Sigma_n$, then Com $f_n$ is in $\Sigma_n$.

## 2. The cohomology groups relative to $c$

### *2.1. The category* Tors$(c, A)$

Now let $A$ be a $c$-trivial abelian group in $\mathbb{V}$ ($c(A) = 1$) and let $L\mathbb{V}[A]$ be the category of left $A$-actions in $\mathbb{V}$.

**Proposition 10.** *If $(V', \upsilon')$ is a left $A$-object and $f : V \to V'$ is a c-Cartesian morphism, then there is a unique left $A$-action $\upsilon$ on $V$ such that $f$ is equivariant. If $(V', \upsilon')$ is c-principal (resp. a c-torsor), then $(V, \upsilon)$ is c-principal (resp. a c-torsor).*

**Proof.** Let us consider the following diagram:

$$\begin{tikzcd}
A\times V \arrow[r, dashed, shift left, "\upsilon"] \arrow[r, shift right, "P_\upsilon"'] \arrow[d, "A\times f"'] & V \arrow[r, "\eta V"] \arrow[d, "f"'] & dcV \arrow[d, "dcf"] \\
A\times V' \arrow[r, dashed, shift left, "\upsilon'"] \arrow[r, shift right, "P_{\upsilon'}"'] & V' \arrow[r, "\eta V'"'] & dcV'.
\end{tikzcd}$$

The two preceding squares being pullbacks, there is a unique $\upsilon : A\times V \to V$ such that $f\cdot \upsilon = \upsilon' \cdot A\times f$, obviously satisfying the left action axioms. Moreover, if $(V', \upsilon')$ is $c$-principal (that is, the lower row is a kernel pair), then the upper row is a kernel pair and $(V, \upsilon)$ is $c$-principal. If furthermore $V'$ has a global $c$-support, then since $f$ is $c$-Cartesian, $V$ has a global $c$-support. □

**Corollary.** *The category $L\mathbb{V}[A]$ admits pullbacks along c-Cartesian equivariant morphisms.* □

**Proposition 11.** *Every equivariant morphism f between two c-torsors is c-Cartesian.*

**Proof.** The internal functor $F_1(f): F_1(V,\upsilon) \to F_1(V', \upsilon')$ determined by $f$ is a discrete fibration since $f_1(V', \upsilon')$ and $F_1(f)\cdot f_1(V',\upsilon')$ (which is $f_1(V,\upsilon)$) are discrete fibrations. Now $(V,\upsilon)$ and $(V',\upsilon')$ being $c$-principal, $F_1(f)$ is equal to $G_1(f): G_1(V)\to G_1(V')$ and, as a discrete fibration between $c$-discrete equivalence relations, is $\pi_0$-Cartesian [6]. Furthermore, $V$ and $V'$ having global $c$-supports, $f$ is $c$-Cartesian. □

**Remark.** This result is the fibered version of the well-known classical result according to which an equivariant map between two ordinary torsors is invertible.

## *2.2. The categories* Tors$(c, A)$ *and* $A$-Cat$_c$

Let $A$ be an abelian group in $\mathbb{V}$. Let us denote by $\mathrm{DF}/K_1A$ the category of discrete fibrations over $K_1A$ and by $\mathrm{Grd}_c\mathbb{V}/K_1A$ the usual category of morphisms of $\mathrm{Grd}_c\mathbb{V}$ with codomain $K_1A$. The inclusion $\mathrm{DF}/K_1A \to \mathrm{Grd}_c\mathbb{V}/K_1A$ determines an embedding $j: L\mathbb{V}[A]\to \mathrm{Grd}_c\mathbb{V}/K_1A$ which has a left adjoint $\varphi_1$, given by the canonical final-discrete fibration factorization

$$\begin{tikzcd}
Y_1 \arrow[dr, "\varphi_1(f_1)"'] & & X_1 \arrow[ll, "\psi_1"'] \arrow[dl, "f_1"] \\
& K_1A. &
\end{tikzcd}$$

Moreover, if $X_1 = G_1X$, then ($\psi_1$ being final) we have $\psi_1 = G_1\psi$ and $Y_1 = G_1Y$ with $\psi$ $c$-invertible, $X$ and $Y$ having the same $c$-support. In particular, if $X$ has a global $c$-support, then $Y$ has a global $c$-support. So let us denote by $A$-Cat$_c$ the full subcategory of $G_1/K_1A$ whose objects $(U, u_1)$, called $A$-categories, are such that $U$ has a global $c$-support. Then the following restriction of the previous embedding (again denoted by the same symbol):

$$j:\mathrm{Tors}(c,A)\to A\text{-}\mathrm{Cat}_c$$

admits a left adjoint (again denoted by $\varphi_1$).

**Consequence.** The underlying set of the group $H^1(c,A)$ can be equally described as the connected components of $\mathrm{Tors}(c,A)$ and as the connected components of $A$-$\mathrm{Cat}_c$. The following section will be devoted to the investigation of a monoidal structure on $A$-$\mathrm{Cat}_c$, giving rise to the same group structure on $H^1(c,A)$. It will appear much simpler than the one on $\mathrm{Tors}(c,A)$.

**Proposition 12.** *The functor $\varphi_1 : A\text{-}\mathrm{Cat}_c \to \mathrm{Tors}(c,A)$ is a fibered reflection, whose $\varphi_1$-Cartesian morphisms $f:(U,u_1)\to(V,v_1)$ are such that $f$ is c-Cartesian and whose $\varphi_1$-invertible morphisms are such that $f$ is c-invertible.*

**Proof.** Let us consider the following diagram, where the square is a pullback and $g_1$ a discrete fibration:

$$\begin{tikzcd}
X_1 \arrow[d, "\psi_1"'] & T_1 \arrow[l, "\bar h_1"'] \arrow[d, "\bar\psi_1"] \\
Y_1 \arrow[d, "\varphi_1(f_1)"'] & Z_1 \arrow[l, "h_1"'] \arrow[dl, "g_1"] \\
K_1A. &
\end{tikzcd}$$

Then $h_1$ is a discrete fibration, thus $\bar\psi_1$ is final (since $\psi_1$ is final) and $g_1=\varphi_1(g_1\cdot\bar\psi_1)$. Now if $X_1=GX$, $Y_1=G_1Y$, $Z_1=G_1Z$, $X$, $Y$, $Z$ having global $c$-supports, then $\psi_1=G_1\psi$, $h_1=G_1h$, $\bar\psi_1=G_1\bar\psi$ and $\bar h_1=G_1 h$. Moreover, $h$, being a morphism in $\mathrm{Tors}(c,A)$, is $c$-Cartesian. So $\bar h$ is $c$-Cartesian and $T$ has a global $c$-support. Thus $\bar h:(T,g_1\cdot G_1\bar\psi)\to(X,f_1)$ is the Cartesian map above $h:(Z,g_1)\to(Y,\varphi_1(f_1))$.

Now a morphism $k:(U,u_1)\to(V,v_1)$ is $\varphi_1$-Cartesian if and only if the following square is a pullback:

$$\begin{tikzcd}
G_1U \arrow[rr, "G_1k"] \arrow[d, "G_1\psi"'] & & G_1V \arrow[d, "G_1\chi"] \\
G_1\bar U \arrow[rr, "G_1\bar k"] \arrow[dr, "\varphi_1(u_1)"'] & & G_1\bar V \\
& K_1A. \arrow[ur, "\varphi_1(v_1)"'] &
\end{tikzcd}$$

But $\bar{k}$, being a morphism of $c$-torsors, is $c$-Cartesian. Then, if the square is a pullback, $k$ is $c$-Cartesian. Conversely, if $k$ is $c$-Cartesian, then the square is a pullback, being the image by $G_1$ of a square in $\mathbb{V}$ having the pair of parallel edges $(k, \bar{k})$ $c$-Cartesian, and the pair $(\psi, \chi)$ $c$-invertible.

The morphism $k$ in $A$-$\mathrm{Cat}_c$ is $\varphi_1$-invertible when $\bar{k}$ is invertible. Now if $\psi$ and $\chi$ are $c$-invertible, then $k$ is $c$-invertible. Conversely, if $k$ is $c$-invertible, then $G_1 k$ is final and the canonical decomposition of $u_1$ is, up to isomorphism, $\varphi_1(v_1) \cdot G_1(\chi \cdot k)$. Consequently, $k$ is $\varphi_1$-invertible. $\square$

**Remark.** The previous terminology and notation concerning $A$-$\mathrm{Cat}_c$ come from the following fact: when $\mathbb{V}$ is the category Set of sets and $\mathbb{W}$ is $\mathbb{1}$, an abelian group can be trivially viewed as a discrete category endowed with a closed monoidal structure. Now an $A$-category in the previous sense is nothing but a non-empty category enriched in this closed monoidal structure.

### 2.3. *The symmetric monoidal structures on* $\mathrm{Tors}(c, A)$ *and* $A$-$\mathrm{Cat}_c$

If $A$ is an abelian group in a left exact category $\mathbb{V}$, then $\mathbb{V}/A$ has a canonical monoidal symmetric structure whose unit is $0: 1 \to A$ and the tensor product is given by the following formula: if $(U, u)$ and $(V, u)$ are in $\mathbb{V}/A$, then $(U, u) \otimes (V, v)$ is $(U \times V, + \cdot u \times v)$:

$$\begin{tikzcd} U \arrow[dr, "u"'] & & V \arrow[dl, "v"] & U \times V \arrow[d, "u \times v"] \\ & A & & A \times A. \arrow[ll, "+"'] \end{tikzcd}$$

This tensor product is clearly associative and symmetric ($A$ being abelian).

Furthermore, for each $(U, u)$, the following commutative diagram:

$$\begin{tikzcd} U \times U \arrow[d, "u \times -u"'] & U \arrow[l, "\Delta"'] \arrow[r] & 1 \arrow[dl, "0"] \\ A \times A \arrow[r, "+"] & A & \end{tikzcd}$$

determines two morphisms in $\mathbb{V}/A$,

$$(U, u) \otimes (U, -u) \leftarrow (U, 0) \rightarrow (1, 0).$$

Now in the basic situation and when $A$ is a $c$-trivial abelian group, the functor $d$ being left exact, the previous tensor product is stable on $d/A$.

Let us denote by $H^0(c, A)$ the group of connected components of $d/A$.

In the same way the category $G_1/K_1(A)$ has a tensor product, which is stable on $A$-$\mathrm{Cat}_c$ since the objects with a global $c$-support are stable under product.

Let us now consider a monoidal category $\mathbb{D}$ with unit $I$ and tensor product $\otimes$, and a category $\mathbb{C}$ with an embedding $j:\mathbb{C}\to\mathbb{D}$ having a left adjoint $\varphi$. We shall denote by $\eta: 1_{\mathbb{D}}\to j\cdot\varphi$ the natural transformation.

**Proposition 13.** *If, for each pair $(D, D')$ of objects of $\mathbb{D}$ the morphism $\varphi(\eta D\otimes\eta D')$ is an isomorphism, then there is a canonical monoidal structure on $\mathbb{C}$, defined by $C\otimes C'=\varphi(jC\otimes jC')$ and having $J=\varphi(I)$ as unit.*

**Proof.** Let us sketch for instance the associativity axiom:

$$\begin{aligned} C\otimes(C'\otimes C'') &= \varphi(jC\otimes j(C'\otimes C''))=\varphi(jC\otimes j\varphi(jC'\otimes jC''))\\ &\simeq \varphi(j\varphi jC\otimes j\varphi(jC'\otimes jC''))\simeq\varphi(jC\otimes(jC'\otimes jC''))\\ &\simeq \varphi((jC\otimes jC')\otimes jC''). \qquad \square \end{aligned}$$

A monoidal functor between two monoidal categories $(\mathbb{D},\otimes)$ and $(\mathbb{D}',\otimes')$ is a functor $f:\mathbb{D}\to\mathbb{D}'$ together with a morphism $\nu_I: I'\to f(I)$ and a natural transformation $\nu$,

$$\nu_{D,D'}: f(D)\otimes' f(D')\to f(D\otimes D')$$

satisfying the obvious coherence conditions. It is called strict when $\nu_I$ and $\nu$ are isomorphisms.

It is then clear that $\varphi$ together with

$$\varphi(\eta D\otimes\eta D')^{-1}:\varphi D\otimes\varphi D'\to\varphi(D\otimes D')$$

is a strict monoidal functor and that $j$ together with

$$\eta(jC\otimes jC'): jC\otimes jC'\to j(C\otimes C')$$

is a monoidal functor.

Now taking $j$ to be the following functor:

$$j:\mathrm{Tors}(c,A)\to A\text{-}\mathrm{Cat}_c$$

we have the following corollary:

**Corollary.** *There is, on* $\mathrm{Tors}(c,A)$, *a monoidal structure such that $\varphi_1$ is a strict monoidal functor.*

**Proof.** Let

$$G_1X\xrightarrow{G_1\psi}G_1Y\xrightarrow{\varphi_1(f_1)}K_1(A)$$

and

$$G_1S\xrightarrow{G_1\chi}G_1T\xrightarrow{\varphi_1(g_1)}K_1(A)$$

be the canonical decompositions of $f_1$ and $g_1$, Now let us consider the following diagram:

$$\begin{tikzcd}
G_1(Y\times T) \arrow[d, equal] & G_1(X\times S) \arrow[l, "G_1(\psi\times\chi)"'] \arrow[d, equal] \\
G_1Y\times G_1T \arrow[dr, "\varphi_1(f)\times\varphi_1(g)"'] & G_1X\times G_1S \arrow[l, "G_1\psi\times G_1\chi"] \arrow[d, "f_1\times g_1"] \\
& K_1A\times K_1A.
\end{tikzcd}$$

The decomposition being stable under products, $G_1(\psi\times\chi)$ is final and consequently is $\varphi_1$-invertible. □

**Remark.** A simple diagram chasing argument shows that this tensor product on $\mathrm{Tor}(c,A)$ coincides with the usual tensor product of left $A$-objects, as defined, say, in [1]. In particular $\varphi_1(1,0)=(A,+)$, the canonical action on $A$ by itself.

**Consequence.** The group $H^1(c,A)$ can be equally described as the group of connected components of $\mathrm{Tors}(c,A)$ and as the group of connected components of $A$-$\mathrm{Cat}_c$.

*2.4. The functor $K_1$ and the unit of the monoidal category* $\mathrm{Tors}(c,A)$

Before going further, let us say a little more about $K_1(A)$. When $A$ is $c$-trivial, it is clear that $K_1A$ is the cokernel in $\mathrm{Ab}(\mathrm{Grd}_c\mathbb{V})$ of the inclusion $\kappa_1A:\mathrm{dis}\,A\rightarrowtail G_1A$:

$$0\longrightarrow \mathrm{dis}\,A \underset{\kappa_1A}{\rightarrowtail} G_1A \underset{\varepsilon_1A}{\twoheadrightarrow} K_1A\longrightarrow 0.$$

This is a straight definition for $K_1A$, which is immediately seen to be a $c_0$-trivial abelian group. Now the kernel of $\varepsilon_1A$ being discrete, $\varepsilon_1A$ is a discrete fibration, and, $A$ having obviously a global $c$-support, $\varepsilon_1A$ is thus associated with some $c$-torsor. On the other hand, the following diagram commutes:

$$\begin{tikzcd}
G_1A \arrow[dr, "\varepsilon_1A"'] & & G_11=1 \arrow[ll, "G_1(0)"'] \arrow[dl, "0"] \\
& K_1(A) &
\end{tikzcd}$$

and $0:1\to A$ being $c$-invertible, $G_1(0)$ is final. Consequently, $\varepsilon_1A$ is $\varphi_1(1,0)$.

On the other hand, the construction $K_1$ clearly extends to a functor from the category $\mathrm{Ab}_c(\mathbb{V})$ of $c$-trivial abelian groups in $\mathbb{V}$ to the category $\mathrm{Ab}_{c_0}(\mathrm{Grd}_c\mathbb{V})$ of $c_0$-trivial abelian groups $\mathrm{Grd}_c\mathbb{V}$.

**Proposition 14.** *The functor $K_1$ is additive. Furthermore, it is an equivalence of categories* $\mathrm{Ab}_c(\mathbb{V})\to\mathrm{Ab}_{c_0}(\mathrm{Grd}_c\mathbb{V})$.

**Proof.** The additivity property is clear. Now let $A_1$ be a $c_0$-trivial abelian group in $\mathrm{Grd}_c\mathbb{V}$. Then $A_0=1$. The inverse equivalence is given by $mA_1$ which is clearly a $c$-trivial abelian group in $\mathbb{V}$ since $A_1$ is in $\mathrm{Grd}_c\mathbb{V}$. □

*2.5. The usual first cohomology group*

Let $\mathbb{E}$ be a left exact and Barr-exact category, with a given abelian group $A$ in $\mathbb{E}$. This abelian group $A$ is $e$-trivial with respect to the following basic situation:

$$\mathbb{E} \underset{1}{\overset{e}{\rightleftarrows}} \mathbb{1}.$$

It is clear that $H^0(e,A)=\mathbb{E}(1,A)$ is the usual $H^0(\mathbb{E},A)$. The group $H^1(e,A)$ described by means of $A$-torsors is just the usual $H^1(\mathbb{E},A)$.

Now let us consider the functor $(\ )_0 : \mathrm{Grd}\,\mathbb{E} \to \mathbb{E}$. It is a fibration whose fibers are left exact and Barr-exact. The abelian group $K_1(A)$ is in the fiber above 1. Given an object $X$ in $\mathbb{E}$, it determines by change of map along the terminal map $X \to 1$ an abelian group $X^*(K_1(A))$ in the fiber above $X$. This group is nothing but $K_1(A)\times G_1X$. Now a global element of this group in the fiber above $X$ is simply a functor

$$G_1X \to K_1(A)\times G_1X$$

whose second projection is necessarily the identity. Thus

$$H^0((\ )_0[X], X^*(K_1(A))) = \mathrm{Grd}\,\mathbb{E}(G_1X, K_1A);$$

this determines clearly a functor $\mathbb{E}^{\mathrm{op}} \to \mathrm{Ab}$.

Let $\mathrm{gl}\,\mathbb{E}$ denote the full subcategory of $\mathbb{E}$ whose objects have a global support and by $\theta$ the restriction of the preceding functor $\theta : (\mathrm{gl}\,\mathbb{E})^{\mathrm{op}} \to \mathrm{Ab}$.

**Proposition 15** ($H^1$ as a colimit of $H^0$'s). *The group $H^1(\mathbb{E},A)$ is the colimit of $\theta$.*

**Proof.** Let $U : \mathrm{Ab} \to \mathrm{Set}$ denote the forgetful functor.

Then $A$-$\mathrm{Cat}_c$ is nothing but the Grothendieck category associated with $U\cdot\theta$. Consequently, $H^1(E,A)=\pi_0(A\text{-}\mathrm{Cat}_c)$ is, as a set, the colimit of $U\cdot\theta$.

The following commutative diagram:

$$\begin{tikzcd}
G_1X \arrow[r, "G_1\Delta"] \arrow[d, "g_1+h_1"'] & G_1X\times G_1X \arrow[d, "g_1\times h_1"] \\
K_1A & K_1A\times K_1A \arrow[l, "+_1"]
\end{tikzcd}$$

insures us that the projections

$$\theta(X) \to H^1(\mathbb{E},A)$$

are group homomorphisms, for every $X$ in $\mathrm{gl}\,\mathbb{E}$.

Finally, the existence of products in gl $\mathbb{E}$ gives us a connectivity property in $(\mathrm{gl}\ \mathbb{E})^{\mathrm{op}}$ which implies that $H^1(\mathbb{E},A)$ is actually a colimit in Ab. $\square$

*2.6. The group extension functor*

If $h: A \to A'$ is a group homomorphism in $\mathbb{V}$, then the functor $\mathbb{V}/h: \mathbb{V}/A \to \mathbb{V}/A'$ is clearly a strict monoidal functor.

When $A$ and $A'$ are $c$-trivial, the restriction of $\mathrm{Grd}_c\mathbb{V}/K_1h$ to $A$-$\mathrm{Cat}_c$ factors through $A'$-$\mathrm{Cat}_c$. We shall denote it by $h$, for short:

$$h: A\text{-}\mathrm{Cat}_c \to A'\text{-}\mathrm{Cat}_c.$$

It is obviously a discrete fibration.

On the other hand, let us denote by $\mathrm{Tors}(c,h)$ the composite $\varphi_1 \cdot h \cdot j$. It is the usual group extension functor

$$\mathrm{Tors}(c,h): \mathrm{Tors}(c,A) \to \mathrm{Tors}(c,A').$$

**Proposition 16.** *The functor* $\mathrm{Tors}(c,h)$ *reflects isomorphisms. It is, up to equivalence, a fibration. (These two conditions mean that* $\mathrm{Tors}(c,h)$ *is, up to equivalence, a discrete fibration.)*

**Proof.** Let us consider the following pullback, where $x_1$ and $x_1'$ are discrete fibrations, $X$ and $X'$ have a global $c$-support:

$$\begin{tikzcd}
G_1X \arrow[rr, "G_1\psi"] \arrow[dr, "G_1\mu"] \arrow[dd, "x_1"'] & & G_1Y \arrow[dr, "G_1\nu"] \arrow[dd] & \\
& G_1X' \arrow[rr, "G_1\psi'"] \arrow[dl, "x_1'"] & & G_1Y' \arrow[dl, "\varphi_1(K_1h\cdot x_1')"] \\
K_1A \arrow[rr, "K_1h"'] & & K_1A' &
\end{tikzcd}$$

and where the vertical unlabelled morphism is $\varphi_1(K_1h \cdot x_1)$.

Now $x_1$ and $x_1'$ being discrete fibrations, $\mu$ is $c$-Cartesian, the same holding for $\nu$. By definition of $\varphi_1$, the morphisms $\psi$ and $\psi'$ are $c$-invertible. Consequently, the following square is a pullback:

$$\begin{tikzcd}
X \arrow[r, "\psi"] \arrow[d, "\mu"'] & Y \arrow[d, "\nu"] \\
X' \arrow[r, "\psi'"'] & Y'.
\end{tikzcd}$$

So when $\nu$ is an isomorphism, such is $\mu$.

Let us now consider the following commutative diagram:

$$\begin{tikzcd} G_1X \arrow[r, "G_1\psi"] \arrow[d, "x_1"'] & G_1Y \arrow[d] \arrow[dr, "G_1\tau"] & \\ & & G_1Z \arrow[dl, "z_1"] \\ K_1A \arrow[r, "K_1h"'] & K_1A' & \end{tikzcd}$$

where the vertical unlabelled map is $\varphi_1(K_1h\cdot x_1)$, $z_1$ is a discrete fibration, and $X$ and $Z$ have a global $c$-support. Thus $\psi$ is $c$-invertible and $\tau$ $c$-Cartesian. Now, according to the first part of this proposition, a morphism in $\mathrm{Tors}(c,A)$ above $\tau$ is necessarily given by the following pullback:

$$\begin{tikzcd} X \arrow[r, "\psi"] & Y \\ T \arrow[u, "\bar{\tau}"] \arrow[r, "\bar{\psi}"'] & Z. \arrow[u, "\tau"'] \end{tikzcd}$$

But $\bar{\psi}$ is then $c$-invertible and $\bar{\tau}$ $c$-Cartesian. Then $T$ has a global $c$-support, $x_1\cdot G_1\bar{\tau}$ is a discrete fibration and $\varphi_1(K_1h\cdot x_1\cdot G_1\bar{\tau})$ is, up to isomorphism, equal to $z_1$. $\square$

A direct proof that the group extension functor is a strict monoidal functor is given by a glance at the following diagram:

$$\begin{tikzcd} G_1(X\times X') \arrow[r, "G_1(\psi\times\psi')"] \arrow[d, equal] & G_1(Y\times Y') \arrow[d, equal] \\ G_1(X)\times G_1(X') \arrow[r, "G_1(\psi)\times G_1(\psi')"'] \arrow[d, "x_1\times x_1'"'] & G_1(Y)\times G_1(Y') \arrow[d, "\varphi_1(K_1h\cdot x_1)\times\varphi_1(K_1h\cdot x_1')"] \\ K_1A\times K_1A \arrow[r, "K_1h\times K_1h"] \arrow[d, "+_1"'] & K_1A'\times K_1A' \arrow[d, "+_1"] \\ K_1A \arrow[r, "K_1h"] & K_1A' \end{tikzcd}$$

since the lower square commutes and since $G_1(\psi\times\psi')$ is final, the final-discrete fibration factorization system being stable under products.

**Proposition 17.** *The functors $\varphi_1$ are natural up to isomorphism (i.e. pseudo-natural). That is, the following square commutes up to isomorphism:*

$$\begin{tikzcd} A\text{-}\mathrm{Cat}_c \arrow[r, "h"] \arrow[d, "\varphi_1"'] & A'\text{-}\mathrm{Cat}_c \arrow[d, "\varphi_1"] \\ \mathrm{Tors}(c,A) \arrow[r, "\mathrm{Tors}(c,h)"'] & \mathrm{Tors}(c,A'). \end{tikzcd}$$

**Proof.** Let us consider the following diagram:

$$\begin{tikzcd} G_1X \arrow[r, "G_1(\psi)"] \arrow[d, "x_1"'] & G_1Y \arrow[r, "G_1(\tau)"] \arrow[dl, "\varphi_1(x_1)"] & G_1Z \arrow[dl, "\varphi_1(K_1h\cdot\varphi_1x_1)"] \\ K_1A \arrow[r, "K_1h"'] & K_1A'. & \end{tikzcd}$$

Then $G_1\psi$ and $G_1\tau$ are final and

$$\varphi_1(K_1h\cdot x_1)=\varphi_1(K_1h\cdot\varphi_1(x_1)). \qquad \square$$

**Consequence.** The functors $\Pi_0(h)$ and $\Pi_0(\mathrm{Tors}(c,h))$ determine the same group homomorphism

$$H^1(c,h):H^1(c,A)\to H^1(c,A')$$

*2.7. The connecting functor*

Given an exact sequence of $c$-trivial abelian groups in $\mathbb{V}$,

$$0\longrightarrow A\overset{k}{\rightarrowtail} B\overset{h}{\twoheadrightarrow} C\longrightarrow 0,$$

there is determined, as usual, a $c$-invertible exact diagram in $\mathbb{V}$,

$$A\times A\times B \overset{\longrightarrow}{\underset{\longrightarrow}{\longrightarrow}} A\times B \underset{p_B}{\overset{b}{\rightrightarrows}} B \overset{h}{\twoheadrightarrow} C,$$

where $b=+\cdot[k\cdot p_A,p_B]$ is the canonical action of $A$ on $B$.

Let $R_1[h]$ denote the $c$-discrete equivalence relation given by the left-hand portion of the previous diagram. We shall define the connecting functor $\delta$ in the following way:

$$\delta:d/C\to\mathrm{Tors}(c,A);$$

given $(W,\upsilon)$ in $d/C$, let us consider the following diagram where the square $(*)$ is a pullback:

$$\begin{tikzcd} A\times X \arrow[r, shift left, "x"] \arrow[r, shift right, "p_x"'] \arrow[d, "A\times\xi"'] & X \arrow[r] \arrow[d, "\xi"] \arrow[dr, phantom, "(*)"] & dW \arrow[d, "\upsilon"] \\ A\times B \arrow[r, shift left, "b"] \arrow[r, shift right, "p_b"'] & B \arrow[r, "h"'] & C \end{tikzcd}$$

and let us set $\delta(W,\upsilon)=(X,x)$ which is, as usual, an $A$-left object. The lower line being a kernel pair, so is the upper line and $(X,x)$ is $c$-principal. Furthermore, $h$ being a $c$-invertible regular epimorphism, so is $\sigma$. Then $\sigma$ being, up to isomorphism, equal to $X\to dcX$, $X$ has a global $c$-support and $(X,x)$ is a $c$-torsor.

Let us now give a direct construction in $\mathrm{Grd}_c\mathbb{V}$:

The $c$-discrete equivalence $R_1[h]$, being the kernel equivalence of $h$, the left-hand square of the following diagram is a pullback: (The right-hand square is a pullback by definition of $K_1C$.)

$$\begin{tikzcd}
R_1[h] \arrow[r, "h_1"] \arrow[d, "\beta_1"'] & \mathrm{dis}\, C \arrow[r] \arrow[d, tail] & 1 \arrow[d, tail, "0"] \\
G_1B \arrow[r, "G_1h"'] & G_1C \arrow[r, "\varepsilon_1 C"'] & K_1C.
\end{tikzcd}$$

Now in the following diagram the right-hand square is again a pullback:

$$\begin{tikzcd}
R_1[h] \arrow[r, dashed, "\varrho_1"] \arrow[d, "\beta_1"'] \arrow[rr, bend left] & K_1A \arrow[r] \arrow[d, "K_1k"] & 1 \arrow[d, "0"] \\
G_1B \arrow[r, "\varepsilon_1 B"'] & K_1B \arrow[r, "K_1h"'] & K_1C
\end{tikzcd}$$

and $\varrho_1$ is the unique map making the left-hand square commutative. The two global squares are equal. Consequently, the left-hand square of the second diagram is again a pullback. Thus $\varrho_1$ is a discrete fibration since $\varepsilon_1 B$ is a discrete fibration.

**Remark.** We shall denote by $k_1: G_1A\to R_1[h]$ the unique map such that $h_1\cdot k_1=0$ and $\beta_1\cdot k_1=G_1(k)$. It is a kernel map of $h_1$. We shall denote by $\sigma_1:\mathrm{dis}\, B\to R_1[h]$ the unique map such that $\varrho_1\cdot\sigma_1=0$ and $\beta_1\cdot\sigma_1=\kappa_1 B$. It is a kernel map of $\varrho_1$.

The functor $\delta$ is actually given by the $A$-torsor underlying the left vertical discrete fibration determined by the following construction, the upper square being a pullback:

$$\begin{tikzcd}
\delta_1(W,\upsilon) \arrow[r] \arrow[d] & \mathrm{dis}\, dW \arrow[d, "\mathrm{dis}\,\upsilon"] \\
R_1[h] \arrow[r, "h_1"'] \arrow[d, "\varrho_1"'] & \mathrm{dis}\, C \\
K_1A. &
\end{tikzcd}$$

Let $\mathrm{Tors}_c/R_1[h]$ denote the full subcategory of $G_1/R_1[h]$ whose objects $(U,u_1)$, with $u_1: G_1U \to R_1[h]$, are such that $U$ has a global support and $u_1$ is a discrete fibration. The previous construction is thus the composite of two functors,

$$d/C \xrightarrow[\psi_h]{} \mathrm{Tors}_c/R_1[h] \xrightarrow[\mathrm{Tors}_c/\varrho_1]{} \mathrm{Tors}_c/K_1A.$$

**Proposition 18.** *The functor $\psi_h$ is an equivalence of categories.*

**Proof.** The functor $\psi_h$ is fully faithful. Indeed, let $(W,v)$ and $(W',v')$ be two objects in $d/C$, and let $f_1$ be a morphism making the following diagram commutative:

$$\begin{tikzcd} \delta_1(W,v) \arrow[rr, "f_1"] \arrow[dr] & & \delta_1(W',v') \arrow[dl] \\ & R_1[h]. & \end{tikzcd}$$

Now $\delta_1(W,v)$ and $\delta_1(W',v')$ are equivalence relations whose quotients are $dW$ and $dW'$. Furthermore, $f_1$ is necessarily a discrete fibration. Then, there is a unique morphism $\tau: dW \to dW'$ making the following diagram a pullback:

$$\begin{tikzcd} \delta_1(W,v) \arrow[r] \arrow[d, "f_1"'] & \mathrm{dis}\, dW \arrow[d, "\mathrm{dis}\,\tau"] \\ \delta_1(W',v') \arrow[r] & \mathrm{dis}\, dW'. \end{tikzcd}$$

The functor $\psi_h$ is essentially surjective. Indeed, given an object $(U,u_1)$ in $\mathrm{Tors}_c/R_1[h]$, $u_1: G_1U \to R_1[h]$ is a discrete fibration between equivalence relations. Thus $u_1$ is $\pi_0$-Cartesian (see [6, Proposition 4]), so its factorization $\pi_0 u_1$ between their quotients $dcU$ and $C$ is such that the following diagram is a pullback:

$$\begin{tikzcd} G_1(U) \arrow[r] \arrow[d, "u_1"'] & \mathrm{dis}\, dcU \arrow[d, "\mathrm{dis}\,\pi_0 u_1"] \\ R_1[h] \arrow[r] & \mathrm{dis}\, C. \end{tikzcd} \qquad \square$$

**Remark.** The exactness property of the long cohomology sequence is based, modulo some connectedness property in $n$-$\mathrm{Tors}(c,A)$, upon the above proposition.

On the other hand, the functor $\mathrm{Tors}_c/\varrho_1$ being a discrete fibration, the functor $\delta$ is, up to equivalence, a discrete fibration.

**Proposition 19.** *The functor $\delta$ is a strict monoidal functor.*

**Proof.** The exact sequence of $c$-trivial abelian groups being clearly preserved by the functor $G_1$, the left-hand square of the following diagram is a pullback:

$$\begin{tikzcd}
G_1A \arrow[r] \arrow[d] \arrow[rr, bend left, "G_1k"] & R_1[h] \arrow[r, tail] \arrow[d, "h_1"] & G_1B \arrow[d, "G_1h"] \\
1 \arrow[r, "\mathrm{dis}\,0"'] & \mathrm{dis}\,C \arrow[r, tail] & G_1C
\end{tikzcd}$$

and $\delta(1,0)=(A,+)$.

Let us now show that

$$\delta(c,1)\otimes\delta(c,1)\cong\delta((c,1)\otimes(c,1)).$$

To do so, let us remark that the following diagram commutes, since we are working in $\mathrm{Ab}(\mathrm{Grd}_c\mathbb{V})$:

$$\begin{tikzcd}
R_1[h]\times R_1[h] \arrow[r, "\varrho_1\times\varrho_1"] \arrow[d, "+_1"'] & K_1A\times K_1A \arrow[d, "+_1"] \\
R_1[h] \arrow[r, "\varrho_1"'] & K_1A.
\end{tikzcd}$$

Secondly, let us consider the following diagram in $\mathrm{Ab}(\mathrm{Grd}_c\mathbb{V})$:

$$\begin{tikzcd}
R_1[h]\times R_1[h] \arrow[dr, dashed, "\tau_1"] \arrow[drr, "h_1\times h_1"] \arrow[ddr, "+_1"'] & & \\
& \delta_1((c,1)\otimes(c,1)) \arrow[r] \arrow[d, "\sigma_1"] & \mathrm{dis}\,C\times\mathrm{dis}\,C \arrow[d, "\mathrm{dis}\,+"] \\
& R_1[h] \arrow[r, "h_1"'] & \mathrm{dis}\,C.
\end{tikzcd}$$

By definition the square is a pullback, whence we have a factorization

$$\tau_1: R_1[h]\times R_1[h]\to\delta_1((c,1)\otimes(c,1))$$

which is final since it is a morphism between equivalence relations having the same quotient $C\times C$. Consequently, the canonical decomposition of $+_1\cdot\varrho_1\times\varrho_1=\varrho_1\cdot +_1$ is given by $(\varrho_1\cdot\sigma_1)\cdot\tau_1$ and thus

$$\delta(c,1)\otimes\delta(c,1)\rightrightarrows\delta((c,1)\otimes(c,1)).$$

Now, given two objects $(W, \upsilon)$ and $(W', \upsilon')$ in $d/C$, the following total square is a pullback by definition of $\delta_1(W, \upsilon)$ and $\delta_1(W', \upsilon')$ and so is the right-hand square by definition of $\delta_1((W, \upsilon) \otimes (W', \upsilon'))$:

$$\begin{tikzcd}
\delta_1(W,\upsilon) \times \delta_1(W',\upsilon') \arrow[rr] \arrow[dr, "\bar{\tau}_1"'] \arrow[dd] & & \operatorname{dis} dW \times \operatorname{dis} dW' \arrow[dd, "\operatorname{dis} \upsilon \times \operatorname{dis} \upsilon'"] \\
& \delta_1((W,\upsilon) \otimes (W',\upsilon')) \arrow[ur] \arrow[d, "\chi_1"] & \\
R_1[h] \times R_1[h] \arrow[r, "\tau_1"'] & \delta_1((c,1) \otimes (c,1)) \arrow[r] & \operatorname{dis} C \times \operatorname{dis} C.
\end{tikzcd}$$

Consequently, the left-hand square is a pullback. Now $\chi_1$ is a discrete fibration (since $\operatorname{dis} \upsilon \times \operatorname{dis} \upsilon'$ is a discrete fibration). Thus, $\tau_1$ being final, so is $\bar{\tau}_1$. Whence the result. □

*Naturality of the connecting functor*

Let the following diagram be a transformation between short exact sequences of $c$-trivial abelian groups:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "k"] \arrow[d, "\alpha"'] & B \arrow[r, "h"] \arrow[d, "\beta"'] & C \arrow[r] \arrow[d, "\gamma"'] & 0 \\
0 \arrow[r] & A' \arrow[r, "k'"'] & B' \arrow[r, "h'"'] & C' \arrow[r] & 0.
\end{tikzcd}$$

**Proposition 20.** *The following square commutes up to isomorphism*:

$$\begin{tikzcd}
d/C \arrow[r, "\delta"] \arrow[d, "d/\gamma"'] & \operatorname{Tors}(c, A) \arrow[d, "\operatorname{Tors}(c,\alpha)"] \\
d/C' \arrow[r, "\delta'"'] & \operatorname{Tors}(c, A').
\end{tikzcd}$$

**Proof.** Note that the given transformation yields a commutative square:

$$\begin{tikzcd}
R_1[h] \arrow[r, "{R_1[\beta,\gamma]}"] \arrow[d, "\varrho_1"'] & R_1[h'] \arrow[d, "\varrho_1'"] \\
K_1 A \arrow[r, "K_1 \alpha"'] & K_1 A'.
\end{tikzcd}$$

Now consider the following diagram:

$$\begin{tikzcd}
\delta_1(W,\gamma\cdot\upsilon) \arrow[rrr, "w_1'"] \arrow[dr, phantom, ""] \arrow[ddr] & & & R_1[h'] \arrow[dd] \\
& \delta_1(W,\upsilon) \arrow[ul, dashed, "\tau_1"'] \arrow[r, "w_1"] \arrow[d] & R_1[h] \arrow[ur, "{R_1[\beta,\gamma]}"] \arrow[d] & \\
& \operatorname{dis} dW \arrow[r, "\operatorname{dis}\upsilon"'] & \operatorname{dis} C \arrow[r, "\operatorname{dis}\gamma"'] & \operatorname{dis} c'.
\end{tikzcd}$$

The exterior and the left-hand squares are pullbacks by definition. Whence there exists a factorization $\tau_1$, which is final as a functor between equivalence relations having the same quotient $dW$. Consequently, $w_1'$ is the discrete fibration associated to $R_1[\beta,\gamma]\cdot w_1$. Whence the result. □

## 2.8. *The functor $K_n$ and the higher order cohomology groups*

The $c$-trivial abelian group $A$ in $\mathbb{V}$ yields a $c_{n-1}$-trivial abelian group $K_n(A)$ in $n$-$\mathrm{Grd}_c\mathbb{V}$, by the following formula: $K_n(A)=K_1(K_{n-1}A)$ or, equivalently, by the following cokernel:

$$0 \longrightarrow \operatorname{dis}(K_{n-1}A) \xrightarrow[\kappa_n A]{} G_n(K_{n-1}A) \xrightarrow[\varepsilon_n A]{} K_n(A) \longrightarrow 0$$

**Definition 21.** An $(n+1)$-torsor on $A$ is a $c_{n-1}$-torsor $X_n$ on $K_nA$ such that $X_n$ is $c$-aspherical. We shall denote by $(n+1)$-$\mathrm{Tors}(c,A)$ the full subcategory of $\mathrm{Tors}(c_{n-1},K_nA)$ whose objects are the $(n+1)$-torsors.

*The $n+1$-torsors and the $A_n$-categories*

In the same way as at level 1, the final-discrete fibration factorization in $(n+1)$-$\mathrm{Grd}_c\mathbb{V}$ will give us an alternative description of the group $H^{n+1}(c,A)$.

**Definition 22.** Let us denote by $A_n$-$\mathrm{Cat}_c$ the full subcategory of $G_{n+1}/K_{n+1}(A)$ whose objects $(X_n,x_{n+1})$ (with $x_{n+1}:G_{n+1}X_n\to K_{n+1}A$) are such that $X_n$ is $c$-aspherical. Such objects are called $A_n$-categories.

Now if $(X_n,x_{n+1})$ is a $A_n$-category, then the canonical decomposition

$$\begin{tikzcd}
G_{n+1}Y_n \arrow[dr, "\varphi_1(x_{n+1})"'] & & G_{n+1}X \arrow[ll, "G_{n+1}\psi_n"'] \arrow[dl, "x_{n+1}"] \\
& K_{n+1}A &
\end{tikzcd}$$

is such that $\psi_n$ is $c_{n-1}$-invertible and if $X_n$ has a global $c_{n-1}$-support, then $Y_n$ has a global $c_{n-1}$-support. Consequently, if $X_n$ is aspherical, then $Y_n$ is aspherical and $\varphi_1(x_{n+1})$ determines an $(n+1)$-torsor. Thus the following embedding:

$$j:(n+1)\text{-Tors}(c,A)\to A_n\text{-cat}_c$$

has a left adjoint we shall denote by $\varphi_{n+1}$.

Since the aspherical $c$-discrete $n$-groupoids are stable under products, the tensor product of $G_{n+1}/K_{n+1}A$ is stable on $A_n$-Cat$_c$. From the definition of $\varphi_{n+1}$, we can now conclude that the tensor product of Tors$(c_{n-1},K_nA)$ is stable on $(n+1)$-Tors$(c,A)$, and that $\varphi_{n+1}$ is a strict monoidal functor and a fibered reflection.

**Definition 23.** We shall denote the group of connected components of the monoidal category $(n+1)$-Tors$(c,A)$ by $H^{n+1}(c,A)$ and call it the $(n+1)$th cohomology group of $c$ with values in $A$. It is equally well defined by the group of connected components of the monoidal category $A_n$-Cat$_c$.

**Proposition 24.** *The group $H^{n+1}(c,A)$ is the colimit of the abelian groups $H^0(c_n[X_n],X_n^*(K_{n+1}A))$ with $X_n$ aspherical.*

**Proof.** The group $H^0(c_n[X_n],X_n^*(K_{n+1}A))$ is the group $(n+1)$-Grd$_c\mathbb{V}(G_{n+1}(X_n),$ $K_{n+1}A)$ and the proof is the same as that of Proposition 15. □

*The group $H^{n+1}$ as the colimit of the $H^n$ of the fibers*

Let $\mathbb{E}$ be a left exact and Barr exact category; we are now going to show that $H^{n+1}(\mathbb{E},A)$ is the colimit of the $H^n((\ )_0[X],X^*(K_1A))$, with $X$ having a global support, that is the colimit of the $H^n$ of the fibers of the following fibration, restricted to the objects of $\mathbb{E}$ having a global support:

$$(\ )_0:\mathrm{Grd}\,\mathbb{E}\to\mathbb{E}.$$

Indeed, given a left exact fibration $c:\mathbb{V}\to\mathbb{W}$, instead of considering the $c$-discrete groupoids, we could have considered a more rigid notion, namely that of internal groupoids in $\mathbb{V}$ such that the images by $c$ of their structural maps are identities (instead of isomorphisms). Let us denote by Grd $c$ the full subcategory of Grd$_c\mathbb{V}$ with such objects. In fact, Grd $c$ is the Grothendieck category associated to the pseudo functor:

$$\mathbb{W}^{\mathrm{op}}\to\mathrm{CAT}$$

associating to each object $W$ in $\mathbb{W}$ the category Grd $c[W]$ of internal groupoids in the fiber $c[W]$. Whence the following commutative diagram, with the horizontal functor a Cartesian embedding:

$$\begin{tikzcd}
\mathrm{Grd}\,c \arrow[rr] \arrow[dr, "(\ )_0"'] & & \mathrm{Grd}_c\mathbb{V} \arrow[dl, "c_0"] \\
& \mathbb{V}. &
\end{tikzcd}$$

In the case $c=(\ )_0$, this functor becomes

$$\zeta_1 : \mathrm{Grd}(\ )_0 \to 2\text{-}\mathrm{Grd}\,\mathbb{E}$$

and is furthermore essentially surjective (given an internal 2-groupoid in $\mathbb{E}$, it is always possible to relabel, up to isomorphism, the object of objects of its structural diagram). It is therefore a Cartesian equivalence. In the same way, the pseudofunctor $\mathbb{E}^{\mathrm{op}} \to \mathrm{CAT}$, associating to each object $X$ in $\mathbb{E}$ the category $(n-1)$-Grd$((\ )_0[X])$ of internal $(n-1)$-groupoids in the fiber $(\ )_0[X]$ has a Grothendieck category denoted by $(n-1)$-Grd$(\ )_0$ with a Cartesian equivalence

$$\zeta_{n-1} : (n-1)\text{-}\mathrm{Grd}(\ )_0 \to n\text{-}\mathrm{Grd}\,\mathbb{E}.$$

**Proposition 25** ($H^{n+1}$ as a colimit of $H^n$'s). *For each left exact and Barr-exact category $\mathbb{E}$ the group $H^{n+1}(\mathbb{E},A)$ is the colimit of the groups $H^n((\ )_0[X], X^*(K_1A))$, with $X$ having a global support.*

**Proof.** Let us consider the pseudofunctor

$$\theta_n : (\mathrm{gl}\,\mathbb{E})^{\mathrm{op}} \to \mathrm{CAT}$$

associating to an object $X$ the category $X^*(K_1A)_{n-1}$-Cat and let us denote by $L$ its associated Grothendieck category. Now it is clear that $K_n(K_1A)=K_{n+1}A$. Thus there is an embedding $\gamma$ making the following diagram commutative:

$$\begin{tikzcd} L \arrow[r, "\gamma"] \arrow[d] & A_n\text{-Cat} \arrow[d] \\ (n-1)\text{-Grd}(\ )_0 \arrow[r, "\zeta_{n-1}"'] & n\text{-Grd}\,\mathbb{E} \end{tikzcd}$$

where the unlabelled arrow is the obvious forgetful functor which associates to every $X^*(K_1A)_{n-1}$-category its underlying internal $(n-1)$-groupoid in $(\ )_0[X]$. This $\gamma$ is actually an equivalence of categories since so is $\xi_{n-1}$.

On the other hand, the functor dis : SET $\to$ CAT has a left adjoint $\Pi_0$ which is actually a left 2-adjoint between the discrete 2-category SET and the 2-category CAT. Now $L$ is the lax colimit of $\theta_n$, which is preserved by the left 2-adjoint $\Pi_0$. Consequently $\Pi_0 L$, as a set, is the colimit of the $\Pi_0 \cdot \theta_n[X] = H^n((\ )_0[X], X^*(K_1A))$. Furthermore, $\Pi_0(\gamma)$ is an isomorphism and, as a set, $H^{n+1}$ is the colimit of the $H^n$. The end of the proof (that it is actually a colimit in Ab) is the same as in Proposition 15. $\square$

**Remark.** In the absolute situation ($\mathbb{E}$ Barr-exact) it is well known that a morphism between two torsors is always invertible. Is there an analogous result at level $n$? Let us briefly point out without detail that a morphism between two $(n+1)$-torsors is an $n$-functor which is always a weak $n$-equivalence (see [8]).

*2.9. The higher-order group extension functors*

Given a group homomorphism $h: A \to A'$ between two $c$-trivial abelian groups, the morphism $K_{n+1}(h): K_{n+1}A \to K_{n+1}A'$ allows us to define, in the same way as at level 1, a functor

$$h: A_n\text{-Cat}_c \to A'_n\text{-Cat}_c$$

which is again a discrete fibration, preserving the tensor product. We have again a functor $(n+1)\text{-Tors}(c,h) = \varphi_{n+1} \cdot H \cdot j$:

$$(n+1)\text{-Tors}(c,h): (n+1)\text{-Tors}(c,A) \to (n+1)\text{-Tors}(c,A').$$

Following the definition of $\varphi_{n+1}$ and the results at level 1, it preserves clearly the tensor product and reflects isomorphisms. It is again a fibration up to isomorphism. To see this, mimicking Proposition 16, let us consider the following pullback:

$$\begin{tikzcd} X_n \arrow[r, "\psi_n"] & Y_n \\ T_n \arrow[u, "\bar{\tau}_n"] \arrow[r, "\psi_n"'] & Z_n \arrow[u, "\tau_n"'] \end{tikzcd}$$

with $\psi_n$ $c_{n-1}$-invertible and $\tau_n$ $c_{n-1}$-Cartesian, $X_n, Y_n, Z_n$ being aspherical. Then $\bar{\tau}_n$ is $c_{n-1}$-Cartesian and thus $T_n$ has a global $c_{n-1}$-support, and $\psi_n$ is $c_{n-1}$-invertible and thus $T_{n-1}$ is aspherical. Consequently, $T_n$ is aspherical.

*2.10. The higher-order connecting functor*

Let $0 \longrightarrow A \xrightarrow{k} B \xrightarrow{h} C \longrightarrow 0$ be an exact sequence of $c$-trivial abelian groups. Then

$$0 \longrightarrow K_nA \xrightarrow{K_nk} K_nB \xrightarrow{K_nh} K_nC \longrightarrow 0$$

is an exact sequence of $c_{n-1}$-trivial abelian groups.

We can now define a higher order connecting functor

$$\delta: C_{n-1}\text{-Cat}_c \to (n+1)\text{-Tors}(c,A)$$

mimicking exactly the construction at level 1 : given $(Y_{n-1}, y_n)$ in $C_{n-1}\text{-Cat}_c$, let us consider the following diagram where the square $*$ is a pullback:

$$\begin{tikzcd} K_nA \times X_n \arrow[r, shift left, "x_n"] \arrow[r, shift right, "p_{X_n}"'] \arrow[d, "K_nA \times \xi_n"'] & X_n \arrow[r, "\sigma_n"] \arrow[d, "\xi_n"'] \arrow[dr, phantom, "(*)"] & G_n(Y_{n-1}) \arrow[d, "y_n"] \\ K_nA \times K_nB \arrow[r, shift left, "b_n"] \arrow[r, shift right, "p_{B_n}"'] & K_nB \arrow[r, "K_nh"'] & K_nC \end{tikzcd}$$

and let us set $\delta(Y_{n-1}, y_n) = (X_n, x_n)$. It is a left $K_nA$-object. Furthermore, $K_nh$ being a $c_{n-1}$-invertible regular epimorphism, so is $\sigma_n$. Now $X_{n-1}$ is isomorphic to $Y_{n-1}$, which implies, on the one hand, that $(X_n, x_n)$ is a $c_{n-1}$-principal left object and that $X_n$ has a global $c_{n-1}$-support, on the other hand, that $X_{n-1}$ is aspherical. Consequently, $X_n$ is aspherical and $(X_n, x_n)$ an $(n+1)$-torsor.

Again there is a direct construction in $(n+1)\text{-Grd}_c\mathbb{V}$: the lower line yields a $c_{n-1}$-discrete equivalence relation $R_{n+1}[h]$, which is the kernel equivalence of $K_nh$. It is given by the following pullback:

$$\begin{tikzcd}
R_{n+1}[h] \arrow[r] \arrow[d, "\beta_{n+1}"'] & \operatorname{dis} K_nC \arrow[d] \\
G_{n+1}K_nB \arrow[r, "G_{n+1}K_nh"'] & G_{n+1}K_nC.
\end{tikzcd}$$

We have again a morphism $\varrho_{n+1} : R_{n+1}[h] \to K_{n+1}A$ making the following square a pullback:

$$\begin{tikzcd}
R_{n+1}[h] \arrow[r, "\varrho_{n+1}"] \arrow[d, "\beta_{n+1}"'] & K_{n+1}A \arrow[d, tail, "K_{n+1}k"] \\
G_{n+1}K_nB \arrow[r, two heads, "\varepsilon_{n+1}B"'] & K_{n+1}B.
\end{tikzcd}$$

Then the functor $\delta$ is given by the $(n+1)$-torsor underlying to the left vertical discrete fibration determined by the following diagram:

$$\begin{tikzcd}
\delta_{n+1}(Y_{n-1}, y_n) \arrow[r] \arrow[d] & \operatorname{dis} G_n(Y_{n-1}) \arrow[d, "\operatorname{dis} y_n"] \\
R_{n+1}[h] \arrow[r] \arrow[d, "\varrho_{n+1}"'] & \operatorname{dis} K_nC \\
K_{n+1}A. &
\end{tikzcd}$$

If we denote by $(n+1)\text{-Tors}_c/R_{n+1}[h]$, the full subcategory of $G_{n+1}/R_{n+1}[h]$ whose objects $(Z_n, z_{n+1})$, with $z_{n+1} : G_{n+1}Z_n \to R_{n+1}[h]$, are such that $Z_n$ is aspherical and $z_{n+1}$ a discrete fibration, then the previous construction is the composite of the two following functors:

$$C_{n-1}\text{-Cat}_c \xrightarrow[\psi_h]{} (n+1)\text{-Tors}_c/R_{n+1}[h] \xrightarrow[(n+1)\text{-Tors}_c/\varrho_{n+1}]{} (n+1)\text{-Tors}_c/K_{n+1}A.$$

Again $\psi_h$ is an equivalence of categories, again $\delta$ is a strict monoidal functor.

Finally, these higher order connecting functors are pseudonatural with respect to transformation of short exact sequences.

*2.11. The long cohomology sequence*

Given a short exact sequence of $c$-trivial abelian groups in $\mathbb{V}$,

$$0 \longrightarrow A \xrightarrow{k} B \xrightarrow{h} C \longrightarrow 0,$$

we have thus a long sequence of group homomorphisms

$$\cdots \longrightarrow H^n(c,A) \xrightarrow{k^*} H^n(c,B) \xrightarrow{h^*} H^n(c,C) \xrightarrow{\partial} H^{n+1}(c,A) \xrightarrow{k^*} H^{n+1}(c,B) \longrightarrow \cdots$$

where, for every $f$, the morphism $f^*$ is $\Pi_0(n\text{-Tors}(c,f))$ or $\Pi_0(f)$ and $\partial = \Pi_0(\delta)$.

That $h^* \cdot k^*$ is zero is a consequence of the fact that $h \cdot k$ is zero.

That $\partial \cdot h^*$ is zero is a consequence of the following proposition:

**Proposition 26.** *Given an exact sequence of abelian groups in* $\mathbb{V}$, *there is a unique morphism* $\lambda_1 : G_1A \times \operatorname{dis} B \to R_1[h]$ *making the two following squares pullbacks*:

$$\begin{tikzcd} G_1A \arrow[d, "\varepsilon_1 A"'] & G_1A \times \operatorname{dis} B \arrow[l, "p_1(A)"'] \arrow[r, "p_1(B)"] \arrow[d, dashed, "\lambda_1"] & \operatorname{dis} B \arrow[d, "\operatorname{dis} h"] \\ K_1A & R_1[h] \arrow[l, "\varrho_1"] \arrow[r, "h_1"'] & \operatorname{dis} C \end{tikzcd}$$

*where* $p_1(A)$ *and* $p_1(B)$ *are the projections.*

**Proof.** The image of the short exact sequence by the functors dis, $G_1$ and $K_1$ gives a 'nine lemma' diagram in the abelian category $\operatorname{Ab}(\operatorname{Grd}_c\mathbb{V})$. Then, provided that $\lambda_1 = G_1A \times \operatorname{dis} B \to R_1[h]$ is just $]k_1, \sigma_1[$, this result is pure diagram chasing. $\square$

Then any pullback of $\operatorname{dis} h \cdot \operatorname{dis} \upsilon$ along $h_1$ factorizes through $\varepsilon_1 A$ and consequently $\partial \cdot h^*$ is zero at level 1. The proof is exactly the same at level $n$.

Finally, that $k^* \cdot \partial$ is zero is a consequence of the following: Consider the commutative diagram

$$\begin{tikzcd} G_1B \arrow[d, "\varepsilon_1 B"'] & R_1[h] \arrow[l, "\beta_1"'] \arrow[r, "h_1"'] \arrow[d, "\varrho_1"] & \operatorname{dis} C \\ K_1B & K_1A. \arrow[l, "K_1k"] \end{tikzcd}$$

The composite of the morphism $K_1k \cdot \varrho_1$ by the pullback of any morphism $\operatorname{dis} \upsilon$ along $h_1$ factorizes through $\varepsilon_1 B$. Consequently, $k^* \cdot \partial$ is zero at level 1. The proof is obviously the same at level $n$.

## 3. The exactness property of the long cohomology sequence

### 3.1. *The connectedness of* $n$-Tors$(c,A)$

In order to prove the exactness of this long sequence, we need first to study the connected components of $n$-Tors$(c,A)$.

The aim of the following paragraph is to show that in any category of the form $n$-Tors$(c,A)$, if there are two morphisms

$$X \xrightarrow{f} V \xleftarrow{g} Y,$$

then there are two morphisms:

$$X \xleftarrow{f'} Z \xrightarrow{g'} Y.$$

Consequently, two objects will be in the same connected component if and only if the second situation is satisfied.

Unfortunately, if $n>1$, the category $n$-Tors$(c,A)$ does not admit pullbacks in general and therefore we must find another method. This will be to exhibit, in any category $n$-Tors$(c,A)$ and for any object $X$ in this category, what we could call a universal co-unit interval or a cohomotopy system, that is, an object Coh $X$ and a pair of morphisms

$$\mathrm{Coh}\, X \underset{\omega}{\overset{\alpha}{\rightrightarrows}} X.$$

Then the requirement concerning the connectedness will be obtained by the following pullback which will always exist:

$$\begin{tikzcd} Z \arrow[r, "\psi"] \arrow[d, "{[f',g']}"'] & \mathrm{Coh}\, V \arrow[d, "{[\alpha,\omega]}"] \\ X\times Y \arrow[r, "f\times g"'] & V\times V. \end{tikzcd}$$

These cohomotopy systems are all the more interesting as, when $\mathbb{E}=\mathbb{A}$ is abelian they are exchanged by the new denormalization equivalences, with the universal classifiers of chain homotopies.

*The connected components of* Tors$(c,A)$

There is no problem with the category Tors$(c,A)$ since it admits pullbacks. Indeed, given two equivariant morphisms $(X,x)\xrightarrow{f}(V,v)\xleftarrow{g}(Y,y)$ between $c$-torsors, the morphism $f$ being certainly $c$-Cartesian, there is a pullback in $L\mathbb{V}[A]$:

$$\begin{tikzcd} (Z,z) \arrow[r, "f'"] \arrow[d, "g'"'] & (Y,y) \arrow[d, "g"] \\ (X,x) \arrow[r, "f"'] & (V,v). \end{tikzcd}$$

The morphism $f'$ is again $c$-Cartesian and $(Z,z)$ is a $c$-torsor since $(Y,y)$ is a $c$-torsor. Here, at level 1, the cohomology system is reduced for each object $V$ to

$$V \underset{1}{\overset{1}{\rightrightarrows}} V.$$

*The connected components of* 2-Tors$(c,A)$

The previous proof fails at level 2. Indeed, if $(Z_1,z_1)$ denotes the analogous of $(Z,z)$ in a similar diagram, but with an index 1 everywhere, then $(Z_1,z_1)$ is certainly a $c_0$-torsor, but $c_0(Z_1)=Z_0$ fails to have a global $c$-support in general, since neither $f_0$ nor $g_0$ are necessarily $c$-Cartesian, and $Z_1$ is no more in general $c$-aspherical.

However, we have a cohomotopy system in $\mathrm{Grd}_c\mathbb{V}$, given by

$$\mathrm{Com}\, V_1 \underset{\tau_1}{\overset{\sigma_1}{\rightrightarrows}} V_1.$$

We are going to show that it determines a cohomotopy system on $A_1$-$\mathrm{Cat}_c$ and 2-Tors$(c,A)$.

First, we saw that Com $V_1$ is aspherical when $V_1$ is itself aspherical. Now Com is a left exact functor and extends to a functor

$$\mathrm{COM}: \mathrm{Grd}(\mathrm{Grd}_c\mathbb{V}) \to \mathrm{Grd}(\mathrm{Grd}_c\mathbb{V}).$$

When $V_2$ is in 2-$\mathrm{Grd}_c\mathbb{V}$, COM $V_2$ is no more in 2-$\mathrm{Grd}_c\mathbb{V}$. But $c$ being a left exact fibered reflection, the following embedding:

$$\mathrm{Grd}_c\mathbb{V} \to \mathrm{Grd}\,\mathbb{V}$$

has always a right adjoint $(\check{\ })$ (see [4]), and consequently $\mathrm{COM}\check{\ }\, V_2$ is again in 2-$\mathrm{Grd}_c\mathbb{V}$. Thus the left exact functor Com with the $c_0$-Cartesian natural transformations $\sigma$ and $\tau$ extends to a left exact functor $\mathrm{COM}\check{\ }: 2\text{-}\mathrm{Grd}_c\mathbb{V} \to 2\text{-}\mathrm{Grd}_c\mathbb{V}$ with $c_1$-Cartesian natural transformations $\tilde{\sigma}$ and $\tilde{\tau}$:

$$\mathrm{COM}\check{\ }\, V_2 \underset{\tilde{\tau}_2}{\overset{\tilde{\sigma}_2}{\rightrightarrows}} V_2$$

with $\tilde{\sigma}_2$ and $\tilde{\tau}_2$ $c_1$-Cartesian above $\sigma_1$ and $\tau_1$.

Now if $V_2=G_2V_1$, then $\mathrm{COM}\check{\ }\, V_2$ is just $G_2(\mathrm{Com}\, V_1)$. If $V_2=K_2A$, then $V_1=1$, Com $V_2=1$, and $\mathrm{COM}\check{\ }(K_2A)=K_2A$. Whence, for any object $(V_1,\upsilon_2)$ in $A_1$-$\mathrm{Cat}_c$, the following commutative diagram in 2-$\mathrm{Grd}_c\mathbb{V}$:

$$\begin{array}{ccc} G_2(\mathrm{Com}\,\mathbb{V}_1) & \underset{G_2(\tau_1)}{\overset{G_2(\sigma_1)}{\rightrightarrows}} & G_2V_1 \\ {\scriptstyle \mathrm{COM}\check{\ }\upsilon_2}\downarrow & & \downarrow{\scriptstyle \upsilon_2} \\ K_2A & \underset{1}{\overset{1}{\rightrightarrows}} & K_2A. \end{array}$$

Then $(\mathrm{Com}\, V_1, \mathrm{COM}\check{\ }\, \upsilon_2)$ is an object in $A_1$-$\mathrm{Cat}_c$ (denoted by $\mathrm{Com}(V_1,\upsilon_2)$ for short) which determines a cohomotopy system in $A_1$-$\mathrm{Cat}_c$:

$$\mathrm{Com}(V_1, v_2) \underset{\tau_1}{\overset{\sigma_1}{\rightrightarrows}} (V_1, v_2).$$

When furthermore $v_2$ is a discrete fibration (that is underlying to a 2-torsor), then $G_2(\sigma_1)$ is a discrete fibration (since $\sigma_1$ is $c_0$-Cartesian) and $\mathrm{Com}(V_1, v_2)$ is underlying to a 2-torsor. Thus the previous cohomotopy system is stable on 2-Tors$(c, A)$.

Now the functor Com being left exact, its extension to $A_1$-$\mathrm{Cat}_c$ is a strict monoidal functor. Finally the natural transformation $\mathrm{Com}(V_1, v_2) \to \mathrm{Com}\, \varphi_2(V_1, v_2)$ yields a natural transformation

$$\varphi_2\, \mathrm{Com}(V_1, v_2) \to \mathrm{Com}\, \varphi_2(V_1, v_2)$$

which makes the extension of Com to 2-Tors$(c, A)$ a monoidal functor.

Now, given $(X_1, x_2) \xrightarrow{f_1} (V_1, v_2) \xleftarrow{g_1} (Y_1, y_2)$ in 2-Tors$(c, A)$, the morphisms $f_1$ and $g_1$ are certainly $c_0$-Cartesian, and thus so is $f_1 \times g_1$. Whence the following pullback in the category of left $K_1(A)$-objects:

$$\begin{tikzcd} (Z_1, z_2) \arrow[r, "\psi_1"] \arrow[d, "{[f_1', g_1']}"'] & \mathrm{Com}(V_1, v_2) \arrow[d, "{[\sigma_1, \tau_1]}"] \\ (X_1, x_2) \times (Y_1, y_2) \arrow[r, "f_1 \times g_1"'] & (V_1, v_2) \times (V_1, v_2). \end{tikzcd} \qquad (*)$$

The internal functor $\psi_1$ is $c_0$-Cartesian and $\mathrm{Com}(V_1, v_2)$ is in Tors$(c_0, K_1 A)$, so $(Z_1, z_1)$ is in Tors$(c_0, K_1 A)$. This $(Z_1, z_1)$ will be a 2-torsor when furthermore $Z_0$ has a global $c$-support. This object $Z_0$ is the vertex of the following pullback in $\mathbb{V}$:

$$\begin{tikzcd} Z_0 \arrow[r, "\psi_0"] \arrow[d, "{[f_0', g_0']}"'] & m V_1 \arrow[d, "{[d_0, d_1]}"] \\ X_0 \times Y_0 \arrow[r, "f_0 \times g_0"'] & V_0 \times V_0. \end{tikzcd} \qquad (**)$$

But the canonical decomposition of $[d_0, d_1]$ is the following:

$$m V_1 \xrightarrow{[d_0, d_1]} V_0 \times_c V_0 \longrightarrow V_0 \times V_0$$

since the above $[d_0, d_1]$ is clearly $c$-invertible and the right-hand morphism is $c$-Cartesian above the diagonal $c V_0 \to c V_0 \times c V_0$.

Now $V_1$ has a global $c_0$-support if and only if this $[d_0, d_1]$ is a $c$-invertible regular epimorphism, thus $[d_0, d_1] : m V_1 \to V_0 \times V_0$ is $c$-full. Then $[f_0', g_0']$ is again $c$-full, and $X_0 \times Y_0$ having a global $c$-support, $Z_0$ has a global $c$-support. Consequently, $(Z_1, z_1)$ is a 2-torsor.

*The connected components of* 3-Tors$(c, A)$

The previous construction is not yet sufficient at level 3. Indeed, let us consider, in the category 2-Grd $\mathbb{V}$, a pullback similar to the pullback $(*)$, with objects indexed by 2 instead of 1. There is on $Z_2$ a structure of $c_1$-torsor. Now $Z_1$ appears in a square similar to (**) but indexed by 1 instead of 0. Thus, if $V_2$ has a global $c_1$-support, then again $[d_0, d_1]$ is $c_0$-full and $Z_1$ has a global $c_0$-support. But we cannot conclude in general that $Z_0$, which is given by the following pullback, has a global $c$-support and therefore that $Z_2$ is $c$-aspherical:

$$\begin{tikzcd}
Z_0 \arrow[r] \arrow[d] & V_0 \arrow[d] \\
X_0 \times Y_0 \arrow[r, "f_0 \times g_0"'] & V_0 \times V_0.
\end{tikzcd}$$

In order to overcome this obstruction, we need the following construction which will allow an iteration process to work:

**Definition 27.** Given an object $V_2$ in 2-Grd $\mathbb{V}$, let us call cohomotopy 2-groupoid associated to $V_2$ the object of 2-Grd$_c\mathbb{V}$ defined by the following pullback:

$$\begin{tikzcd}
\mathrm{Coh}_2\, V_2 \arrow[r, "\sigma_2'"] \arrow[d, "\tau_2'"'] & \mathrm{Com}\, V_2 \arrow[d, "\tau_2"] \\
\mathrm{COM}^{\sim}\, V_2 \arrow[r, "\tilde{\sigma}_2"] & V_2.
\end{tikzcd}$$

It determines a cohomotopy system

$$\mathrm{Coh}_2\, V_2 \underset{\omega_2}{\overset{\alpha_2}{\rightrightarrows}} V_2$$

where $\alpha_2$ is the $c_1$-Cartesian map $\sigma_2 \cdot \sigma_2'$ and $\omega_2$ the $c_1$-Cartesian map $\tilde{\tau}_2 \cdot \tau_2'$.

The functor $\mathrm{Coh}_2$, as a pullback of left exact functors, is itself left exact.

**Example.** In the case $\mathbb{V} = \mathrm{Set}$ and $\mathbb{W} = \mathbb{1}$, if $V_2$ is an ordinary 2-groupoid, $\mathrm{Coh}_2\, V_2$ is, up to isomorphism, the groupoid whose objects are the 1-morphisms of $V_2$, whose 1-morphisms are the following squares:

$$\begin{tikzcd}
\cdot \arrow[r] \arrow[d] & \cdot \arrow[d] \arrow[dl, phantom, "\swarrow"] \\
\cdot \arrow[r] & \cdot
\end{tikzcd}$$

and whose 2-morphisms are pairs of coherent 2-morphisms between such 1-morphisms. That is the 2-category of quintets in [16], which classifies the pseudonatural transformations with codomain $V_2$.

**Proposition 28.** *When $V_2$ has a global $c_1$-support (resp. is $c$-aspherical) then*

*$\mathrm{Coh}_2\, V_2$ has a global $c_1$-support (resp. is c-aspherical). Moreover, the morphism $c_1[\alpha_2, \omega_2] : c_1(\mathrm{Coh}_2\, V_2) \to V_1 \times V_1$ is $c_0$-full (resp. in $\Sigma_1$).*

**Proof.** The morphism $\alpha_2$ being $c_1$-Cartesian and $V_2$ having a global $c_1$-support, $\mathrm{Coh}_2\, V_2$ has a global $c_1$-support. Let us now consider the following commutative diagram:

$$\begin{tikzcd}
c_1(\mathrm{Coh}_2\, V_2) \arrow[r, "\sigma_1'"] \arrow[d, dashed, "\bar{\tau}_1"] \arrow[dd, bend right=60, "\tau_1'"'] & c_1(\mathrm{Com}\, V_2) = m V_2 \arrow[d, "{[d_0, d_1]}"] \\
\mathrm{Com}\, V_1 \times_{c_0} \mathrm{Com}\, V_1 \arrow[r, "\sigma_1 \times_{c_0} \sigma_1"'] \arrow[d, "q_1"] & V_1 \times_{c_0} V_1 \arrow[d, "q_1"] \\
\mathrm{Com}\, V_1 \arrow[r, "\sigma_1"'] & V_1
\end{tikzcd}$$

where the $q_1$ denote the second projections. The total square is a pullback as the image by $c_1$ of a pullback. The lower square is a pullback since the vertical edges are $c_1$-invertible and the horizontal ones are $c_1$-Cartesian. Whence there exists a morphism $\bar{\tau}_1$ making the upper square a pullback. Now $V_2$ has a global $c_1$-support if and only if $[d_0, d_1]$ is a ($c_1$-invertible) regular epimorphism. Thus $\bar{\tau}_1$ is a $c_1$-invertible regular epimorphism. Consequently, the canonical decomposition of $c_1[\alpha_2, \omega_2]$ is the following:

$$\begin{tikzcd}
c_1(\mathrm{Coh}_2\, V_2) \arrow[d, "\bar{\tau}_1"'] & & \\
\mathrm{Com}\, V_1 \times_{c_0} \mathrm{Com}\, V_1 \arrow[r] & \mathrm{Com}\, V_1 \times \mathrm{Com}\, V_1 \arrow[r, "\sigma_1 \times \tau_1"] & V_1 \times V_1
\end{tikzcd}$$

and then $C_1[\alpha_2, \omega_2]$ is $c_0$-full.

Now $c_0 \cdot c_1[\alpha_2, \omega_2] = [d_0, d_1] : m V_1 \to V_0 \times V_0$. So, when $V_0$ is aspherical, $[d_0, d_1]$ is $c$-full and $c_1[\alpha_2, \omega_2]$ in $\Sigma_1$. Consequently, $c_1(\mathrm{Coh}_2\, V_2)$ is aspherical. Thus $V_2$ aspherical implies $\mathrm{Coh}_2\, V_2$ aspherical. $\square$

We are now going to show that this cohomotopy system can be extended from $2\text{-}\mathrm{Grd}_c\mathbb{V}$ to $A_2\text{-}\mathrm{Cat}_c$ and $3\text{-}\mathrm{Tors}(c, A)$. The functor $\mathrm{Coh}_2$ being left exact extends to a functor $\mathrm{COH}_2 : \mathrm{Grd}(2\text{-}\mathrm{Grd}_c\mathbb{V}) \to \mathrm{Grd}(2\text{-}\mathrm{Grd}_c\mathbb{V})$. Exactly as it is the case at level 2, we then construct a left exact functor

$$\mathrm{COH}_2^{\sim} : 3\text{-}\mathrm{Grd}_c\mathbb{V} \to 3\text{-}\mathrm{Grd}_c\mathbb{V}$$

with $c_2$-Cartesian natural transformations $\tilde{\alpha}_3$ and $\tilde{\omega}_3$ above $\alpha_2$ and $\omega_2$:

$$\mathrm{COH}_2^{\sim}\, V_3 \underset{\tilde{\omega}_3}{\overset{\tilde{\alpha}_3}{\rightrightarrows}} V_3.$$

Now if $V_3 = G_3 V_2$, then $\mathrm{COH}_2^{\sim} V_3$ is just $G_2(\mathrm{Coh}_2\, V_2)$. If $V_3 = K_3 A$, then $V_2 = 1$ and

$\mathrm{COH}_2^{\sim} V_3 = K_3 A$. Whence, for any object $(V_2, \upsilon_3)$ in $A_2$-Cat$_c$, the following commutative diagram in 3-Grd$_c$ $V$:

$$\begin{tikzcd} G_3(\mathrm{Coh}_2 V_2) \arrow[r, shift left, "G_3(\alpha_2)"] \arrow[r, shift right, "G_3(\omega_2)"'] \arrow[d, "\mathrm{COH}_2^{\sim} \upsilon_3"'] & G_3 V_2 \arrow[d, "\upsilon_3"] \\ K_3 A \arrow[r, shift left, "1"] \arrow[r, shift right, "1"'] & K_3 A. \end{tikzcd}$$

Then $(\mathrm{Coh}_2 V_2, \mathrm{COH}^{\sim} V_3)$ is an object in $A_2$-Cat$_c$ (denoted by $\mathrm{Coh}_2(V_2, \upsilon_3)$ for short) which determines a cohomotopy system in $A_2$-Cat$_c$,

$$\mathrm{Coh}_2(V_2, \upsilon_3) \overset{\alpha_2}{\underset{\omega_2}{\rightrightarrows}} (V_2, \upsilon_3).$$

When furthermore $\upsilon_3$ is a discrete fibration, $G_3(\alpha_2)$ being a discrete fibration, $\mathrm{Coh}_2(V_2, \upsilon_3)$ is underlying a 3-torsor and the previous cohomotopy system is stable on 3-Tors$(c, A)$. Again $\mathrm{Coh}_2$ is a strict monoidal functor on $A_2$-Cat$_c$ and a monoidal functor on 3-Tors$(c, A)$.

Now, given $(X_2, x_3) \xrightarrow{f_2} (V_2, \upsilon_3) \xleftarrow{g_2} (Y_2, y_3)$ in 3-Tors$(c, A)$, the morphisms $f_2$ and $g_2$ are $c_1$-Cartesian, and so is $f_2 \times g_2$. Whence the following pullback in the category of left $K_2(A)$-objects:

$$\begin{tikzcd} (Z_2, z_3) \arrow[r, "\psi_2"] \arrow[d, "{[f_2', g_2']}"'] & \mathrm{Coh}_2(V_2, \upsilon_3) \arrow[d, "{[\alpha_2, \omega_2]}"] \\ (X_2, x_3) \times (Y_2, y_3) \arrow[r, "f_2 \times g_2"'] & (V_2, \upsilon_3) \times (V_2, \upsilon_3). \end{tikzcd}$$

Now $\psi_2$ is $c_1$-Cartesian and $\mathrm{Coh}_2(V_2, \upsilon_3)$ is in Tors$(c_1, K_2 A)$, so $(Z_2, z_3)$ is in Tors$(c_1, K_2 A)$. This $(Z_2, z_3)$ will be a 3-torsor when furthermore $Z_1$ is aspherical. Now $Z_1$ is the vertex of the following pullback:

$$\begin{tikzcd} Z_1 \arrow[r, "\psi_2"] \arrow[d, "{[f_1', g_1']}"'] & c_1(\mathrm{Coh}_2 V_2) \arrow[d, "{c_1[\alpha_2, \omega_2]}"] \\ X_1 \times Y_1 \arrow[r, "f_1 \times g_1"'] & V_1 \times V_1. \end{tikzcd}$$

When $V_2$ is aspherical, $c_1[\alpha_2, \omega_2]$ is in $\Sigma_1$. So $[f_1', g_1']$ is in $\Sigma_1$. The product $X_1 \times Y_1$ being aspherical, so is $Z_1$.

*The connected components of* $(n+1)$-Tors$(c, A)$

Let us suppose we have defined a cohomotopy system in $n$-Grd$_c \mathbb{V}$,

$$\mathrm{Coh}_n V_n \overset{\alpha_n}{\underset{\omega_n}{\rightrightarrows}} V_n,$$

satisfying the following conditions:

(1) the functor $\mathrm{Coh}_n$ is left exact;

(2) the natural transformations $\alpha_n$ and $\omega_n$ are $c_{n-1}$-Cartesian;

(3) if $V_n$ is aspherical, then $c_{n-1}[\alpha_n, \omega_n]$ is in $\Sigma_{n-1}$.

Clearly, this implies that if $V_n$ is aspherical, then $\mathrm{Coh}_n\, V_n$ is aspherical.

Now, mimicking exactly what we did in the last section, we are going to construct a cohomotopy system in $(n+1)$-$\mathrm{Grd}_c\mathbb{V}$ satisfying the same conditions.

First, the functor $\mathrm{Coh}_n$ being left exact, and the natural transformations $\alpha_n$ and $\omega_n$ being $c_{n-1}$-Cartesian, extend to a left exact functor $\mathrm{COH}_n^{\sim}: (n+1)\text{-}\mathrm{Grd}_c\mathbb{V} \to (n+1)\text{-}\mathrm{Grd}_c\mathbb{V}$ and to $c_n$-Cartesian natural transformations $\tilde{\alpha}_{n+1}$ and $\tilde{\omega}_{n+1}$.

**Definition 29.** Given an object $V_{n+1}$ in $(n+1)$-$\mathrm{Grd}_c\mathbb{V}$, let us call cohomotopy $(n+1)$-groupoid associated to $V_{n+1}$, the object defined by the following pullback in $(n+1)$-$\mathrm{Grd}_c\mathbb{V}$:

$$\begin{tikzcd}
\mathrm{Coh}_{n+1}\, V_{n+1} \arrow[r, "\alpha'_{n+1}"] \arrow[d, "\tau'_{n+1}"'] & \mathrm{Com}\, V_{n+1} \arrow[d, "\tau_{n+1}"] \\
\mathrm{COH}_n^{\sim}\, V_{n+1} \arrow[r, "\tilde{\alpha}_{n+1}"'] & V_{n+1}.
\end{tikzcd}$$

The functors Com and $\mathrm{COH}_n^{\sim}$ being left exact, so is $\mathrm{Coh}_{n+1}$. This construction yields a cohomotopy system

$$\mathrm{Coh}_{n+1}\, V_{n+1} \underset{\omega_{n+1}}{\overset{\alpha_{n+1}}{\rightrightarrows}} V_{n+1}$$

where $\alpha_{n+1}$ is the $c_n$-Cartesian morphism $\sigma_{n+1} \cdot \alpha'_{n+1}$ and $\omega_{n+1}$ is the $c_n$-Cartesian morphism $\tilde{\omega}_{n+1} \cdot \tau'_{n+1}$. Finally, the canonical decomposition of $c_n[\alpha_{n+1}, \omega_{n+1}]$ is the following:

$$\begin{tikzcd}
c_n(\mathrm{Coh}_{n+1}\, V_{n+1}) \arrow[d, "\bar{\tau}_n"'] & & \\
\mathrm{Coh}_n\, V_n \times_{n-1} \mathrm{Coh}_n\, V_n \arrow[r] & \mathrm{Coh}_n\, V_n\, \mathrm{Coh}_n\, V_n \arrow[r, "\alpha_n \times \omega_n"] & V_n \times V_n
\end{tikzcd}$$

where $\bar{\tau}_n$ is the last edge of the following pullback:

$$\begin{tikzcd}
c_n(\mathrm{Coh}_{n+1}\, V_{n+1}) \arrow[r, "\alpha'_n"] \arrow[d, "\bar{\tau}_n"'] & c_n(\mathrm{Com}\, V_{n+1}) = m V_{n+1} \arrow[d, "{[d_0, d_1]}"] \\
\mathrm{Coh}_n\, V_n \times_{n-1} \mathrm{Coh}_n\, V_n \arrow[r, "\alpha_n \times_{n-1} \alpha_n"'] & V_n \times_{n-1} V_n.
\end{tikzcd}$$

Furthermore, $c_{n-1} \cdot c_n[\alpha_{n+1}, \omega_{n+1}] = c_{n-1}[\alpha_n, \omega_n]$.

Consequently, if $V_{n+1}$ is aspherical, then $c_n[\alpha_{n+1}, \omega_{n+1}]$ is in $\Sigma_n$.

Now $\mathrm{Coh}_n$ being left exact, $\mathrm{Coh}_n 1 = 1$, and the same reasons, as in the last section, apply to extend this cohomotopy system in the same way to $A_n$-$\mathrm{Cat}_c$ and $(n+1)$-$\mathrm{Tors}(c, A)$.

Finally, the same proof holds for the same result about connectedness in $(n+1)$-$\mathrm{Tors}(c, A)$.

**Remark.** The passage from $\mathrm{Com}\, X_1$ to $\mathrm{Coh}_2\, X_2$ (that is, from 'natural transformations' to 'pseudonatural transformations') is based upon the slogan: "put a 2-morphism wherever there is an equality".

The passage to higher order generalizations is based upon the same slogan: "put a higher order type of morphism wherever there is an equality". This is clearly the meaning of our construction of $\mathrm{Coh}_{n+1}\, V_{n+1}$: by taking the pullback of $\mathrm{Com}\, V_{n+1}$ with $\mathrm{COH}_n^{\sim}\, V_{n+1}$ we add a higher order type of morphism ($\mathrm{Com}\, V_{n+1}$) wherever there was an equality ($\mathrm{COH}_n^{\sim}\, V_{n+1}$).

When $\mathbb{E} = \mathbb{A}$ is an abelian category, the equivalence $N_n$ between $n$-$\mathrm{Grd}\,\mathbb{A}$ and $C^n(\mathbb{A})$ exchanges $\mathrm{Coh}_n(X_n)$ with the universal classifiers of chain homotopies with codomain $N_n(X_n)$.

*The connected component of* 0 *in* $A_n$-$\mathrm{Cat}_c$

Let $(V_n, v_{n+1})$ be an object of $A_n$-$\mathrm{Cat}_c$ lying in the connected component of $(1, 0)$. Then $\varphi_{n+1}(V_n, v_{n+1})$ is in the connected component of $\varphi_{n+1}(1, 0)$ and consequently we are in the following situation:

$$\begin{tikzcd}
(V_n, v_{n+1}) \arrow[d, "\eta_n"'] & (Z_n, z_{n+1}) \arrow[l, dashed, "\bar{\gamma}_n"'] \arrow[d, dashed, "\bar{\eta}_n"] & \\
\varphi_{n+1}(V_n, v_{n+1}) & (X_n, x_{n+1}) \arrow[l, "\gamma_n"] \arrow[r] & (K_n A, \varepsilon_{n+1} A)
\end{tikzcd}$$

with $X_n$ aspherical and $x_{n+1} : G_{n+1}(X_n) \to K_{n+1}A$ a discrete fibration. Let $(Z_n, z_{n+1})$ denote the vertex of the pullback of $\eta_n$ along $\gamma_n$ in $G_{n+1}/K_{n+1}A$. Let us show that $Z_n$ is aspherical. The functor $\gamma_n$ is $c_{n-1}$-Cartesian, thus so is $\bar{\gamma}_n$ and consequently $Z_n$ has a global $c_{n-1}$-support. On the other hand, $\eta_n$ is $c_{n-1}$-invertible, and so is $\gamma_n$. Then $Z_{n-1}$, being isomorphic to $X_{n-1}$, is aspherical, and thus $Z_n$ is aspherical.

Therefore, an object $(V_n, v_{n+1})$ in $A_n$-$\mathrm{Cat}_c$ is in the connected component of 0 if and only if there is an object $(Z_n, z_{n+1})$ in $A_n$-$\mathrm{Cat}_c$ and two morphisms

$$(V_n, v_{n+1}) \leftarrow (Z_n, z_{n+1}) \rightarrow (K_n A, \varepsilon_{n+1} A).$$

*3.2. The exactness property of the long cohomology sequence*

*Exactness at B*

**Proposition 30.** *The following sequence of abelian groups is exact*:
$$H^{n+1}(c,A) \xrightarrow{k^*} H^{n+1}(c,B) \xrightarrow{h^*} H^{n+1}(c,C).$$

**Proof.** Let $(V_n, \upsilon_{n+1})$ be an object in $B_n$-$\mathrm{Cat}_c$ such that $h(V_n, \upsilon_{n+1})$ is in the connected component of 0, and let us consider the following diagram satisfying $\varepsilon_{n+1}C \cdot G_{n+1}(g_n) = K_{n+1}(h) \cdot \upsilon_{n+1} \cdot G_{n+1}(f_n)$ with $V_n \xleftarrow{f_n} Z_n \xrightarrow{g_n} K_nC$:

$$\begin{tikzcd}
 & G_{n+1}(Z_n) \arrow[drr, "G_{n+1}(g_n)"] \arrow[dd] & & \\
G_{n+1}(Y_n) \arrow[ur, dashed, "G_{n+1}(p_n)"] \arrow[rr, dashed, "G_{n+1}(q_n)"] \arrow[ddr, dashed, "y_{n+1}"'] & & G_{n+1}K_nB \arrow[r, "G_{n+1}K_nh"] \arrow[dd, "\varepsilon_{n+1}B"] & G_{n+1}K_nC \arrow[dd, "\varepsilon_{n+1}C"] \\
 & G_{n+1}V_n \arrow[dr, "\upsilon_{n+1}"] & & \\
 & K_{n+1}A \arrow[r, tail, "K_{n+1}k"'] & K_{n+1}B \arrow[r, two heads, "K_{n+1}h"'] & K_{n+1}C
\end{tikzcd}$$

and where $Y_n$ is defined by the following pullback:

$$\begin{tikzcd}
 & Z_n \arrow[ddr, "g_n"] & \\
Y_n \arrow[ur, "p_n"] \arrow[dr, "q_n"'] & & \\
 & K_nB \arrow[r, two heads, "K_nh"'] & K_nC.
\end{tikzcd}$$

Now $K_nh$ is a $c_{n-1}$-invertible regular epimorphism and thus so is $p_n$. So if $Z_n$ is aspherical, then $Y_n$ is again aspherical.

Moreover, we have the following equalities:

$$\begin{aligned}
K_{n+1}(h) \cdot \varepsilon_{n+1}B \cdot G_{n+1}(q_n) &= \varepsilon_{n+1}C \cdot G_{n+1}(K_nh) \cdot G_{n+1}(q_n) \\
&= \varepsilon_{n+1}C \cdot G_{n+1}(g_n) \cdot G_{n+1}(p_n) = K_{n+1}(h) \cdot \upsilon_{n+1} \cdot G_{n+1}(f_n) \cdot G_{n+1}(p_n).
\end{aligned}$$

Consequently,

$$K_{n+1}(h) \cdot [\upsilon_{n+1} \cdot G_{n+1}(f_n \cdot p_n) - \varepsilon_{n+1}B \cdot G_{n+1}(q_n)] = 0.$$

Whence there exists a morphism $y_{n+1} : G_{n+1}(Y_n) \to K_{n+1}A$ such that

$$K_{n+1}(k) \cdot y_{n+1} = \upsilon_{n+1} \cdot G_{n+1}(f_n \cdot p_n) - \varepsilon_{n+1}B \cdot G_{n+1}(q_n)$$

or, equivalently,

$$v_{n+1} \cdot G_{n+1}(f_n \cdot p_n) - K_{n+1}(k) \cdot y_{n+1} = \varepsilon_{n+1}B \cdot G_{n+1}(q_n).$$

This last equality implies that, in $H^{n+1}(c,B)$, we have

$$(\overline{V_n, v_{n+1}}) = k^*(\overline{Y_n, y_{n+1}}).$$

The foregoing proof is only valid for $n>0$. We must also show the exactness of

$$H^0(c,A) \xrightarrow{k^*} H^0(c,B) \xrightarrow{h^*} H^0(c,C).$$

But $d/C$ admits pullbacks and the proof is straightforward. □

*Exactness at C*

**Proposition 31.** *The following sequence of abelian groups is exact*:

$$H^n(c,B) \xrightarrow{h^*} H^n(c,C) \xrightarrow{\partial} H^{n+1}(c,A).$$

**Proof.** Let $(V_{n-1}, v_n)$ be an object in $C_{n-1}$-$\mathrm{Cat}_c$ such that $\delta_n(V_{n-1}, v_n)$ is in the connected component of zero, and let us consider the following diagram with $\varrho_{n+1} \cdot \chi_{n+1} \cdot G_{n+1}(f_n) = \varepsilon_{n+1}A \cdot G_{n+1}(g_n)$:

$$\begin{tikzcd}
 & G_{n+1}Z_n \arrow[dl, "G_{n+1}(g_n)"'] \arrow[d, dashed, "z_{n+1}"] \arrow[rr, "G_{n+1}(f_n)"] & & \delta_{n+1}(V_{n-1}, v_n) \arrow[dl] \arrow[r, "\theta_{n+1}"] & \operatorname{dis} G_n(V_{n-1}) \arrow[ddd, "\operatorname{dis} v_n"] \\
G_{n+1}K_nA \arrow[dd, "\varepsilon_{n+1}A"'] & G_{n+1}K_nA \times \operatorname{dis} K_nB \arrow[l, "p_{n+1}(A)"'] \arrow[dd, "\lambda_{n+1}"] \arrow[rr, "p_{n+1}(B)"] & & \operatorname{dis} K_nB \arrow[dd, "\operatorname{dis} K_nh"] & \\
 & & \chi_{n+1} & & \\
K_{n+1}A & R_{n+1}[h] \arrow[l, "\varrho_{n+1}"] \arrow[rr, "h_{n+1}"'] & & \operatorname{dis} K_n. &
\end{tikzcd}$$

The lower left-hand square being a pullback, there is a unique factorization $z_{n+1}: G_{n+1}Z_n \to G_{n+1}K_nA \times \operatorname{dis} K_nB$, such that $p_{n+1}(A) \cdot z_{n+1} = G_{n+1}(g_n)$ and $\lambda_{n+1} \cdot z_{n+1} = \chi_{n+1} \cdot G_{n+1}(f_n)$.

Consequently, $\operatorname{dis} K_nh \cdot [p_{n+1}(B) \cdot z_{n+1}] = \operatorname{dis} v_n \cdot \theta_{n+1} \cdot G_{n+1}(f_n)$. The image by the functor $\pi_n : (n+1)$-$\mathrm{Grd}_c\mathbb{V} \to n$-$\mathrm{Grd}_c\mathbb{V}$ of this last equation yields a commutative diagram in $n$-$\mathrm{Grd}_c\mathbb{V}$:

$$\begin{tikzcd}
G_nZ_{n-1} \arrow[r, "\pi_n(\theta_{n+1} \cdot G_{n+1}(f_n))"] \arrow[d, "\pi_n(p_{n+1}(B) \cdot z_{n+1})"'] & G_nV_{n-1} \arrow[d, "v_n"] \\
K_nB \arrow[r, "K_nh"'] & K_nC
\end{tikzcd}$$

and consequently we have $h^*(\overline{Z_{n-1}, \pi_n(p_{n+1}(B) \cdot z_{n+1})}) = (\overline{V_{n-1}, v_n})$ in $H^n(c,C)$. □

*Exactness at A*

**Proposition 32.** *The following sequence of abelian groups is exact:*

$$H^n(c,C) \xrightarrow{\partial} H^{n+1}(c,A) \xrightarrow{k^*} H^{n+1}(c,B).$$

**Proof.** Let $(V_n, \upsilon_{n+1})$ be an object in $A_n$-$\mathrm{Cat}_c$ such that $k(V_n, \upsilon_{n+1})$ is in the connected component of 0, and let us consider the following diagram with $K_{n+1}k \cdot \upsilon_{n+1} \cdot G_{n+1}(f_n) = \varepsilon_{n+1}B \cdot G_{n+1}(g_n)$:

$$\begin{tikzcd}
 & R_{n+1}[h] \arrow[r] \arrow[d, "\beta_{n+1}"'] & \operatorname{dis} K_nC \arrow[d, tail, "\kappa_{n+1}C"] \\
G_{n+1}Z_n \arrow[ur, dashed] \arrow[r, "G_{n+1}(g_n)"] \arrow[d, "G_{n+1}(f_n)"'] & G_{n+1}K_nB \arrow[r, "G_{n+1}(K_nh)"] \arrow[dd, "\varepsilon_{n+1}B"] & G_{n+1}K_nC \arrow[dd, "\varepsilon_{n+1}C"] \\
G_{n+1}V_n \arrow[d, "\upsilon_{n+1}"'] & & \\
K_{n+1}A \arrow[r, tail, "K_{n+1}(k)"'] & K_{n+1}B \arrow[r, two heads, "K_{n+1}(h)"'] & K_{n+1}C.
\end{tikzcd}$$

Now $\varepsilon_{n+1}C \cdot G_{n+1}(K_nh \cdot g_n)$, being factored through $K_{n+1}(h \cdot k)$, is zero, and there is a morphism $y_{n+1} : G_{n+1}Z_n \to \operatorname{dis} K_nC$ such that $\kappa_{n+1}C \cdot y_{n+1} = G_{n+1}(K_nh \cdot g_n)$. Whence there exists a morphism $z_{n+1} : G_{n+1}Z_n \to R_{n+1}[h]$ such that $\beta_{n+1} \cdot z_{n+1} = G_{n+1}(g_n)$. It is easy to check ($K_{n+1}(k)$ being a monomorphism) that

$$\varrho_{n+1} \cdot z_{n+1} = \upsilon_{n+1} \cdot G_{n+1}(f_n).$$

Now by Proposition 18, $(Z_n, \varrho_{n+1} \cdot z_{n+1})$ is, up to isomorphism, in the image of $\delta_{n+1}$. Then the previous equation means that $\overline{(V_n, \upsilon_{n+1})}$ is in the image of $\partial$.

The above proof is only valid for $n>0$. Let us show now that the following sequence is exact:

$$0 \longrightarrow H^0(c,A) \xrightarrow{k^*} H^0(c,B).$$

Let $(W, \upsilon)$ be an object of $d/A$ whose image by $k$ is in the connected component of 0. Then, considering the following diagram:

$$\begin{tikzcd}
 & dW' \arrow[dl] \arrow[dr] & \\
dW \arrow[d, "\upsilon"'] & & 1 \arrow[d, "0"] \arrow[dll, "0"'] \\
A \arrow[rr, tail, "k"'] & & B,
\end{tikzcd}$$

we have necessarily $\upsilon \cdot df = 0$ and $\overline{(W, \upsilon)} = 0$ in $H^0(c,A)$. $\square$

## 4. The classical theories

### 4.1. *Yoneda's* $\mathrm{Ext}^n$

Let $\mathbb{A}$ be an abelian category and $C$ an object of $\mathbb{A}$. Then the category $\mathbb{A}/C$, though no longer abelian, is an exact category. An abelian group in $\mathbb{A}/C$ is necessarily given by a projection $A\times C\to C$ with $A$ any object of $\mathbb{A}$. Let $C^*(A)$ denote this abelian group in $\mathbb{A}/C$. We are now going to show that the category $n$-Tors$(\mathbb{A}/C, C^*(A))$ is equivalent to the category $\mathrm{EXT}^n(C,A)$ of $n$-fold extensions of $A$ by $C$ and that consequently the groups $H^n(\mathbb{A}/C, C^*(A))$ and $\mathrm{Ext}^n(C,A)$ are isomorphic.

Let us first recall that there is, for any integer $n$, an equivalence making the following diagram commutative:

$$\begin{tikzcd}
n\text{-Grd}\,\mathbb{A} \arrow[r, shift left, "N_n"] \arrow[d, "(\ )_{n-1}"'] & C^n(\mathbb{A}) \arrow[l, shift left, "D_n"] \arrow[d, "T_{n-1}"] \\
(n-1)\text{-Grd}\,\mathbb{A} \arrow[r, shift left, "N_{n-1}"] & C^{n-1}(\mathbb{A}) \arrow[l, shift left, "D_{n-1}"]
\end{tikzcd}$$

where $C^n(\mathbb{A})$ denotes the category of abelian complexes of length $n$ in $\mathbb{A}$ and $T_{n-1}$ the truncation of the last element of an $n$-complex. This functor $T_{n-1}$ has a right adjoint $\mathrm{Kr}_n$ (which is the augmentation by the kernel) equivalent to the functor $G_n$.

Now, any internal abelian group in $\mathbb{A}$ is reduced to the data of an object $A$ in $\mathbb{A}$. The image of the group $K_nA$ in $n$-Grd $\mathbb{A}$ by the functor $N_n$ is then the following $n$-complex, we shall again, improperly, denote by $K_nA$:

$$A\to 0\to 0\to\cdots\to 0\to 0.$$

The category $\mathbb{A}/C$ is the fiber above $C$ of the fibration $T_0: C^1(\mathbb{A})\to\mathbb{A}$. Consequently, $\mathrm{Grd}(\mathbb{A}/C)$ is equivalent to the category $C^2(\mathbb{A})_c$ of 2-complexes in $\mathbb{A}$ ending with $C$. The functor corresponding to $(\ )_0$, is just the restriction of $T_1: C^2(\mathbb{A})_c\to\mathbb{A}/C$. Its right adjoint, corresponding to $G_1$, is the augmentation of a 1-complex by its kernel.

More generally, according to the new denormalization theorem, the category $n$-Grd$(\mathbb{A}/C)$ is equivalent to the category $C^{n+1}(\mathbb{A})_c$ of $(n+1)$-complexes ending with $C$ whose morphisms are just transformations of $(n+1)$-complexes with $1_C$ at $C$. The functor $(\ )_{n-1}: n\text{-Grd}(\mathbb{A}/C)\to(n-1)\text{-Grd}(\mathbb{A}/C)$ is equivalent to the restriction of $T_n: C^{n+1}(\mathbb{A})_c\to C^n(\mathbb{A})_c$, its right adjoint being again the augmentation of an $n$-complex by its kernel. Therefore, an object of $n$-Grd$(\mathbb{A}/C)$ with a global $(\ )_{n-1}$-support corresponds to an $(n+1)$-complex which is exact at level $n$ and an aspherical object of $n$-Grd$(\mathbb{A}/C)$ corresponds to an $(n+1)$-complex which is exact at any level.

On the other hand, a left action of the group $K_nA$ on an object of $C^n(\mathbb{A})$ is obviously equivalent to an augmentation of this $n$-complex by $A$. This action is $T_{n-1}$-principal if and only if this augmentation is actually a kernel augmentation.

Consequently, the category of left $K_n(C^*A)$-objects in $n$-Grd$(\mathbb{A}/C)$ is equivalent to the category $C^{n+2}(C,A)$ of $(n+2)$-complexes between $A$ and $C$ whose morphisms are transformations of complexes with $1_A$ at $A$ and $1_C$ at $C$. The category $(n+1)$-Tors$(\mathbb{A}/C, C^*(A))$ is then equivalent to the category $\mathrm{EXT}^{n+1}(C,A)$ of $(n+1)$-fold extensions of $A$ by $C$, and the groups $H^{n+1}(C,A)$ and $\mathrm{Ext}^{n+1}(C,A)$ are isomorphic.

### *4.2. Cohomology of groups in the sense of Eilenberg–Mac Lane*

It is now possible to come back to our starting point, namely the cohomology of groups.

The category **Grp** of abstract groups is an exact category. We are now going to show that the interpretation of the cohomology groups of a group, determined by the construction given in this paper, is, up to isomorphism, the same as those given by Holt [22] and Huebschmann [23] by means of crossed $n$-fold extensions.

Let $Q$ be an abstract group. Then the category Grp/$Q$ is again an exact category. Let $(A,z,+)$ be an internal abelian group in Grp/$Q$:

$$Q \underset{z}{\overset{a}{\leftrightarrows}} A \xleftarrow{\ +\ } A\times_Q A.$$

The pullback of $a$ along the map $1\to Q$ determines an abelian group $M=\operatorname{Ker} a$, on which $Q$ is acting through $z$: $q\cdot m = z(q)\cdot m\cdot z(q^{-1})$. Conversely, given a left $Q$-module $M$, the semi-direct product $M\ltimes Q$ determines an abelian group in Grd $Q$. Let us denote by $Q^*(M)$ this abelian group. Then $H^0(\mathrm{Grp}/Q, Q^*(M))$ is nothing but the group of the sections of $M\ltimes Q\to Q$, that is, the group of derivations $\mathrm{Der}(Q,M)$. The aim of this section is to show that the group $H^n(\mathrm{Grp}/Q, Q^*M)$ is isomorphic to the group $\mathrm{Opext}^n(Q,M)$ of [23] (see also [22]).

It is well known that Grd(Grp) is equivalent to the category $X$-Mod of crossed modules, where a crossed module [10] $(C,G,\partial)$ is a pair of groups $(C,G)$, endowed with a left action of $G$ on $C$, written $(g,c)\to {}^gC$ and a homomorphism $\partial: C\to G$ of $G$-groups where $G$ acts on itself by conjugation. Moreover, the map $\partial$ must satisfy $b\cdot c\cdot b^{-1} = {}^{\partial(b)}c$ for each $(b,c)$ in $C\times C$. The notion of morphism is natural. Let $(\ )_0$ be the forgetful functor $X$-Mod$\to$Grp which associates $G$ to $(C,G,\partial)$.

Let $N$: Grd(Grp)$\to X$-Mod denote this equivalence. Clearly $(\ )_0\cdot N = (\ )_0$ and $N\cdot G_1(G) = (G,G,\mathrm{id})$ where $G$ is acting on itself by conjugation. We shall denote again, improperly, this functor $N\cdot G_1$ by $G_1$.

A natural problem, now, is to determine to which category the category 2-Grd(Grp) is equivalent.

**Definition 33** (see for instance [23]). Let us call a crossed 2-fold complex a sequence $C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} G$ of group homomorphisms such that

(1) $(C_1, G, \partial_1)$ is a crossed module;

(2) $C_2$ is a $G$-module and $\partial_2$ a morphism of left $G$-action;

(3) $\partial_1 \cdot \partial_2 = 0$;
(4) ${}^{\partial_1(c_1)}c_2 = c_2$ for each $(c_1, c_2)$ in $C_1 \times C_2$.

This last condition means that $C_2$ is actually a $G/\partial_1\ c_1$-module. A consequence of (2) and (4) is that $\partial_2 C_2$ is in the center of $C_1$. The notion of morphism between two crossed 2-fold complexes is the natural one. Let us denote this category by 2-$X$-Mod.

**Proposition 34.** *The categories* 2-Grd(Grp) *and* 2-$X$-Mod *are equivalent.*

**Proof.** The categories Grd(Grp) and $X$-Mod being equivalent, it is sufficient to prove that the standard construction applied to:

$$\mathrm{Grp} \underset{G_1}{\overset{(\ )_0}{\rightleftarrows}} X\text{-Mod}$$

is equivalent to 2-$X$-Mod. For that let us study what is a ( $)_0$-discrete groupoid: it is a sequence of group homomorphisms,

$$\begin{tikzcd} C_1 \arrow[d, "\partial_1"'] & K_1 \arrow[l, shift left, "d_0"'] \arrow[l, shift right, "d_1"] & K_2 \arrow[l, shift left] \arrow[l, shift right] \\ G & & \end{tikzcd}$$

where $(C_1, G, \partial_1)$ is a crossed module, $\partial_1 \cdot d_0 = \partial_1 \cdot d_1 = \partial$, $(K_1, G, \partial)$ is a crossed module, $d_0$ and $d_1$ are morphisms of crossed modules, $K_2$ is the pullback of $d_0$ along $d_1$ in $X$-Mod.

Let $C_2$ denote $\operatorname{Ker} d_1$ and $\partial_2$ the restriction of $d_0$ to $C_2$. Then clearly, $\partial_1 \cdot \partial_2 = 0$, $C_2$ is a $G$-module and $\partial_2$ a morphism of left action.

Now $k_1 \cdot k_1' \cdot k_1^{-1} = {}^{\partial k_1}k_1'$ for each $(k_1, k_1')$ in $K_1 \times K_1$. If $k_1$ is in $C_2$, then $\partial(k_1) = \partial_1 \cdot d_1(k_1) = 1$ and $C_2$ is in the center of $K_1$. Thus,

$${}^{\partial_1(c_1)}c_2 = {}^{\partial(s_0 c_1)}c_2 = s_0 c_1 \cdot c_2 \cdot s_0 c_1^{-1} = c_2.$$

Furthermore, $K_1$ is isomorphic to $C_2 \times C_1$ by $k_1 \to (k_1 \cdot s_0 d_1(k_1^{-1}), d_1(k_1))$.

Conversely, given a crossed 2-fold complex, let $K_1$ be the product of $C_2$ and $C_1$ as left $G$-objects. Then it is easy to check that, defining $\partial(c_2, c_1)$ as $\partial_1(c_1)$, we get a crossed module $(C_2 \times C_1, G, \partial)$ and, defining $d_1(c_2, c_1)$ as $c_1$ and $d_0(c_2, c_1)$ as $\partial_2(c_2) \cdot c_1$, we get a ( $)_0$-discrete groupoid. $\square$

We thus get a pair of adjoint functors:

$$X\text{-Mod} \underset{G_2}{\overset{(\ )_1}{\rightleftarrows}} 2\text{-}X\text{-Mod}$$

defined by $G_2(C, G, \partial) = (\operatorname{Ker} \partial \longrightarrow C \xrightarrow{\partial} G)$ and $(C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} G)_1 = (C_1, G, \partial_2)$, satisfying the conditions of the basic situation.

More generally,

**Definition 35** (see for instance [23]). Let us call a crossed $n$-fold complex, a sequence

$$C_n \xrightarrow{\partial_n} C_{n-1} \longrightarrow \cdots \longrightarrow C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} G$$

of group homomorphisms such that

(1) $(C_1, G, \partial_1)$ is a crossed module;

(2) for $2 \leq k \leq n$, each $C_k$ is a $Q$-module, where $Q = G/\partial_1 C_1$ and each $\partial_k$ is a $Q$-map;

(3) $\partial_{k-1}\partial_k = 0$.

The notion of morphism of crossed $n$-fold complexes is the natural one. Let $n$-$X$-Mod denote this category.

**Proposition 36.** *The categories* $n$-Grd(Grp) *and* $n$-$X$-Mod *are equivalent.*

**Proof.** By induction, and mimicking exactly the proof of Proposition 34. We thus get the following commutative diagram:

$$\begin{tikzcd}
n\text{-Grd(Grp)} \arrow[r, shift left, "N_n"] \arrow[d, "(\ )_{n-1}"'] & n\text{-}X\text{-Mod} \arrow[l, shift left, "D_n"] \arrow[d, "T_{n-1}"] \\
(n-1)\text{-Grd(Grp)} \arrow[r, shift left, "N_{n-1}"] & (n-1)\text{-}X\text{-Mod} \arrow[l, shift left, "D_{n-1}"]
\end{tikzcd}$$

where the functor $T_{n-1}$ is the truncation of the last element. $\square$

As a consequence, the category $n$-Grd(Grp/$Q$) is equivalent to the category $n$-$X$-Mod$_Q$ whose objects are the sequences of group homomorphisms:

$$C_n \xrightarrow{\partial_n} C_{n-1} \longrightarrow \cdots \longrightarrow C_1 \xrightarrow{\partial_1} G \xrightarrow{\partial} Q$$

where the indexed part is a crossed $n$-fold complex such that

$$\partial \cdot \partial_1 = 0.$$

Now the functor $(\ )_{n-1} : n$-Grd(Grp/$Q$) $\to (n-1)$-Grd(Grp/$Q$) is equivalent to the truncation of the last element, and $G_n$ to the augmentation by the kernel. Thus, an object of $n$-Grd(Grp/$Q$) having a global $(\ )_{n-1}$-support corresponds to a sequence which is exact at level $(n-1)$, and an aspherical object to a sequence which is exact at any level.

On the other hand, given a left $Q$-module $M$, the object of $n$-$X$-Mod$_Q$ corresponding to $K_n(Q^*(M))$ is the following:

$$M \longrightarrow 1 \longrightarrow \cdots \longrightarrow 1 \longrightarrow Q \xrightarrow{\mathrm{Id}} Q.$$

Let us denote it by $K_n(M)$. A left $K_n(M)$-action on an object of $n$-$X$-Mod$_Q$ is obviously equivalent to an augmentation of this object by $M$, the action being $T_{n-1}$-principal if and only if this augmentation is a kernel augmentation. Consequently, an $(n+1)$-torsor corresponds to a crossed $(n+1)$-fold extension of $M$ by $Q$ [23], that is, an exact sequence of groups

$$0 \longrightarrow M \xrightarrow{\gamma} C_n \xrightarrow{\partial_n} C_{n-1} \longrightarrow \cdots \longrightarrow C_1 \xrightarrow{\partial_1} G \xrightarrow{\partial} Q \longrightarrow 1$$

with the following properties:

(1) $(C_1, G, \partial_1)$ is a crossed module;

(2) for $1 < k \leq n$, $C_k$ is a $Q$-module, and the morphisms $\partial_K$ and $\gamma$ are $Q$-linear.

Consequently the category $(n+1)$-Tors(Grd/$Q$, $Q^*(M)$) is equivalent to the category OPEXT$^{n+1}(Q,M)$ of crossed $(n+1)$-fold extensions of $M$ by $Q$, and the groups $H^{n+1}(\mathrm{Grp}/Q, Q^*(H))$ and Opext$^{n+1}(Q,M)$ are isomorphic.

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